

ON THE HILBERT MATRIX, II¹

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1. The Hilbert Matrix is $H_k = ((n + m + 1 - k)^{-1}), m, n = 0, 1, 2, \dots$, where k is a real number that is not a positive integer. It is known [2; 3; 7] that if x_0, x_1, x_2, \dots , is a sequence of complex numbers,² then

$$0 \leq \sum_{n,m=0}^{\infty} (n + m + 1 - k)^{-1} x_n x_m^* \leq M_k \sum_{n=0}^{\infty} |x_n|^2$$

where the best possible constant M_k is π for $k \leq 1/2$ and $\pi |\csc \pi k|$ for $1/2 < k$. Thus, when considered as a linear operator on the complex sequential Hilbert space l_2 , H_k is a bounded symmetric operator. Magnus [8] showed that the l_2 spectrum of H_0 is purely continuous and consists of the interval $[0, \pi]$. In this note we shall exhibit for each real k a monotone function $\rho_k(\lambda)$ and an isometric map V_k of l_2 onto $L^2(d\rho_k)$ such that $V_k H_k V_k^{-1}$ is a multiplication operator. This will allow us to determine the spectral nature of H_k .

In [9] we studied an isomorphism of l^2 with $L^2(0, \infty)$ that transforms the Hilbert operator H_k into an integral operator which we shall now denote by $\mathfrak{H}_{k,1/2}$. It can be easily checked that $\mathfrak{H}_{k,1/2}$ formally commutes with the differential operator L_k which is defined below. Indeed, we shall prove that $\mathfrak{H}_{k,1/2} = \pi \operatorname{sech} \pi L_k^{1/2}$. Since L_k can be diagonalized by a now standard procedure so $\mathfrak{H}_{k,1/2}$ and hence H_k can be diagonalized.

2. We first shall apply the Titchmarsh-Kodaira theory of singular differential operators [11; 4], to the operator L_k , where $(L_k y)(x) = -(x^2 y'(x))' - (kx - x^2/4 + 1/4)y(x)$, $x \geq 0$, k real. Suppose that λ is a complex number with positive imaginary part and $u = i\lambda^{1/2}$, $\pi < \arg u < 3\pi/2$. $L_k y = \lambda y$ has the linearly independent solutions $\alpha_k(x, \lambda) = W_{k,u}(x)x^{-1}$ and $\beta_k(x, \lambda) = [\Gamma(1 - 2u)]^{-1} \Gamma(1/2 - k - u) \cdot M_{k,-u}(x)x^{-1}$, where W and M are Whittaker functions [1, Chapter 6]. Considered as functions of x , $\alpha_k \in L^2(1, \infty)$, $\alpha_k \notin L^2(0, 1)$, $\beta_k \in L^2(0, 1)$, $\beta_k \notin L^2(1, \infty)$, and the Wronskian of α_k and β_k is 1. Thus L_k is of the limit point type at 0 and ∞ and has the Green's

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² We use an asterisk for complex conjugation.

function $G_k(t, s, \lambda) = G_k(s, t, \lambda) = \alpha_k(t, \lambda)\beta_k(s, \lambda)$ if $t \geq s$.

If $\lambda > 0$, then $W_{k,u}^*(x) = W_{k,-u}(x) = W_{k,u}(x)$ and $\beta_k(x, \lambda) - \beta_k^*(x, \lambda) = i\pi^{-1} \sinh(2\pi\lambda^{1/2}) |\Gamma(1/2 - k - u)|^2 \alpha_k(x, \lambda)$. Thus if $t \geq s$ it follows that

$$\begin{aligned} \operatorname{Im} G_k(t, s, \lambda) &= -2^{-1}i[\beta_k(s, \lambda) - \beta_k^*(s, \lambda)]\alpha_k(t, \lambda) \\ &= (2\pi)^{-1} \sinh(2\pi\lambda^{1/2}) |\Gamma(1/2 - k - u)|^2 \alpha_k(s, \lambda)\alpha_k(t, \lambda). \end{aligned}$$

For fixed s and t , G_k is meromorphic in $\operatorname{Re} \lambda < 0$, and the poles of G_k in the λ plane are determined by the poles of $\Gamma(1/2 - k - u)$. Thus if $k < 1/2$, then G_k has no poles. Suppose $k \geq 1/2$. We put $\lambda_{n,k} = -(k - 1/2 - n)^2$, $n = 0, 1, 2, \dots, N_k$, where $N_k \leq k - 1/2$. The residue of $\Gamma(1/2 - k - u)$ at $\lambda_{n,k}$ is $(-1)^n (n!)^{-1} (2n + 1 - 2k)$, so the residue of G_k at $\lambda_{n,k}$ for $t \geq s$ is

$$\begin{aligned} \frac{(-1)^n (2n + 1 - 2k)}{n! \Gamma(2n - 2k)} \beta_k(s, \lambda_{n,k}) \cdot \alpha_k(t, \lambda_{n,k}) \\ = \frac{2n + 1 - 2k}{n! \Gamma(2k - n)} \alpha_k(s, \lambda_{n,k}) \alpha_k(t, \lambda_{n,k}). \end{aligned}$$

Hence, by [11; 4] we have established

THEOREM 1. *Suppose $-\infty < k < \infty$. Let $\rho_k(\lambda)$ be the monotone increasing function $= (1/2\pi^2) \int_0^\lambda \sinh(2\pi\xi^{1/2}) |\Gamma(1/2 - k - i\xi^{1/2})|^2 d\xi$ if $\lambda \geq 0$, $= 0$ if $\lambda < 0$, $k < 1/2$, and $= \sum_{\lambda < \lambda_{n,k}} (2n + 1 - 2k)/n! \Gamma(2k - n)$ if $\lambda < 0$, $k \geq 1/2$. By $L^2(d\rho_k)$ we mean the Hilbert space with norm given by $\|g\| = [\int_{-\infty}^\infty |g(\lambda)|^2 d\rho_k(\lambda)]^{1/2}$.*

Let U_k be the operator on $L^2(0, \infty)$ to $L^2(d\rho_k)$ defined by $(U_k f)(\lambda) = \text{l.i.m.}_{\epsilon \rightarrow 0^+} \int_\epsilon^\infty \alpha_k(x, \lambda) f(x) dx$, $f \in L^2(0, \infty)$, $-\infty < \lambda < \infty$. Then

(i) U_k is an isometric transformation that maps $L^2(0, \infty)$ onto $L^2(d\rho_k)$, so if $f \in L^2(0, \infty)$ and $\underline{g} = U_k f$, then

$$\begin{aligned} \int_0^\infty |f(x)|^2 dx &= \int_{-\infty}^\infty |g(\lambda)|^2 d\rho_k(\lambda) \\ &= \frac{1}{2} \pi^{-2} \int_0^\infty |g(\lambda)|^2 \sinh(2\pi\lambda^{1/2}) \left| \Gamma\left(\frac{1}{2} - k - i\lambda^{1/2}\right) \right|^2 d\lambda \\ &\quad + \sum_{n=0}^{N_k} |g(\lambda_{n,k})|^2 \frac{2k - 2n - 1}{n! \Gamma(2k - n)}. \end{aligned}$$

(ii) For any $g \in L^2(d\rho_k)$, $(U_k^{-1}g)(x) = \int_{-\infty}^\infty \alpha_k(x, \lambda) g(\lambda) d\rho_k(\lambda)$, where the integral is understood to converge in $L^2(d\rho_k)$ norm.

(iii) If $\lambda g(\lambda) \in L^2(d\rho_k)$, then $(U_k L_k U_k^{-1} g)(\lambda) = \lambda g(\lambda)$ except for a set of $d\rho_k$ measure 0.

3. Thus the isometric map U_k diagonalizes L_k . Next we consider a class of integral operators on $L^2(0, \infty)$ that are bounded functions of L_k .

THEOREM 2. Suppose $\text{Re } \gamma > 0$ and $1/2 - k + \gamma \neq 0, -1, -2, \dots$. Let $\mathfrak{C}_{k,\gamma}$ be the operator on $L^2(0, \infty)$ to $L^2(0, \infty)$ defined by

$$(\mathfrak{C}_{k,\gamma} f)(x) = \Gamma(1/2 - k + \gamma) \int_0^\infty (xt)^{\gamma-1/2} (x+t)^{-\gamma-1/2} W_{k,\gamma}(x+t) f(t) dt.$$

Then $\mathfrak{C}_{k,\gamma}$ is a bounded normal operator such that

$$(U_k \mathfrak{C}_{k,\gamma} U_k^{-1} g)(\lambda) = \Gamma(\gamma + i\lambda^{1/2}) \Gamma(\gamma - i\lambda^{1/2}) g(\lambda)$$

except for a set of $d\rho_k$ measure 0.

PROOF. Hari Shanker [10] showed that if $\text{Re } (\gamma \pm u) > 0, 1/2 - k + \gamma \neq 0, -1, -2, \dots$, then $\Gamma(\gamma + u) \Gamma(\gamma - u) W_{k,u}(x) x^{-1} = \Gamma(1/2 + \gamma - k) \int_0^\infty (xt)^{\gamma-1/2} (x+t)^{-\gamma-1/2} W_{k,\gamma}(x+t) W_{k,u}(t) t^{-1} dt$. If $k > 1/2, \text{Re } \gamma > 0, u_n = i\lambda_{n,k}^{1/2}, n = 0, 1, \dots, N_k, N_k < k - 1/2$, then $W_{k,u_n}(t) t^{\gamma-3/2} \in L(0, \infty)$, so in this case the condition $\text{Re } (\gamma \pm u) > 0$ may be replaced by the restriction $\text{Re } \gamma > 0$. Thus if $g(\lambda)$ is continuous with compact support the Fubini theorem assures us that

$$(\mathfrak{C}_{k,\gamma} U_k^{-1} g)(x) = \Gamma(\gamma - i\lambda^{1/2}) \Gamma(\gamma - i\lambda^{1/2}) (U_k^{-1} g)(x).$$

By operating on the left with U_k we obtain

$$(*) \quad (U_k \mathfrak{C}_{k,\gamma} U_k^{-1} g)(\lambda) = \Gamma(\gamma + i\lambda^{1/2}) \Gamma(\gamma - i\lambda^{1/2}) g(\lambda).$$

Since $\Gamma(\gamma + i\lambda^{1/2}) \Gamma(\gamma - i\lambda^{1/2})$ is a.e. bounded with respect to $d\rho_k$ measure, $\mathfrak{C}_{k,\gamma}$ is a bounded operator. Finally, (*) holds for all g in a dense subset of $L^2(d\rho_k)$ and hence for all $g \in L^2(d\rho_k)$.

By specializing to $k = 0$ we obtain a result of Lebedev [5; 6].

COROLLARY 3 (LEBEDEV).

$$(U_0 f)(\lambda) = \text{l.i.m.}_{\epsilon \rightarrow 0+} \pi^{-1/2} \int_\epsilon^\infty K_{\lambda^{1/2}}(x/2) x^{-1/2} f(x) dx$$

provides an isometric map of $L^2(0, \infty)$ onto the Hilbert space with norm given by $\|g\| = [\int_0^\infty |g(x)|^2 \sinh(\pi\lambda^{1/2}) d\lambda]^{1/2}$, with³

$$(U_0^{-1} g)(x) = \pi^{-3/2} \int_0^\infty K_{i\lambda^{1/2}}(x) x^{-1/2} g(\lambda) \sinh(\pi\lambda^{1/2}) d\lambda.$$

³ K_u is the modified Bessel function of the third kind.

Let $(\mathfrak{H}_{0,1/2} f)(x) = \int_0^\infty e^{-(x+y)/2}(x+y)^{-1}f(y)dy$. Then $(U_0\mathfrak{H}_{0,1/2} U_0^{-1}g)(\lambda) = \pi \operatorname{sech}(\pi\lambda^{1/2})g(\lambda) = \pi \operatorname{sech}(\pi L^{1/2})g(\lambda)$.

PROOF. Use $W_{0,u}(x) = \pi^{-1/2}x^{1/2}K_u(x/2)$, $W_{0,1/2}(x) = e^{-x/2}$ and $\Gamma(1/2 - i\lambda^{1/2})\Gamma(1/2 + i\lambda^{1/2}) = \pi \operatorname{sech}(\pi\lambda^{1/2})$.

THEOREM 4. Let $\phi_n(x) = e^{-x/2}L_n(x)$, $n = 0, 1, 2, \dots$, where L_n is the n th Laguerre function. Define the operator V_k on l^2 by specifying that whenever $a = \{a_n\}_0^\infty \in l^2$, then $(V_k a)(\lambda) = U_k(\sum_{n=0}^\infty a_n \phi_n)$. It follows that:

(i) V_k is an isometric map of l^2 onto $L^2(d\rho_k)$ whose inverse V_k^{-1} is given by $V_k^{-1}g = a = \{a_n\}$, where

$$a_n = \int_0^\infty (U_k^{-1}g)(x)\phi_n(x)dx, \quad n = 0, 1, 2, \dots$$

(ii) If $g \in L^2(d\rho_k)$, and k is not a positive integer, then $(V_k H_k V_k^{-1}g)(\lambda) = (U_k \mathfrak{H}_{k,1/2} U_k^{-1}g)(\lambda) = \pi \operatorname{sech}(\pi\lambda^{1/2})g(\lambda)$ except for a set of $d\rho_k$ measure zero.

PROOF. (i) is true since U_k is isometric and the ϕ_n form a complete orthonormal set in $L^2(0, \infty)$. (ii) is a consequence of the relation $\int_0^\infty (\mathfrak{H}_{k,1/2}\phi_n)(x)\phi_m(x)dx = (n+m+1-k)^{-1}$, $n, m = 0, 1, 2, \dots$, proved in [9] for $k < 1$ and easily seen valid for all $k \neq 1, 2, 3, \dots$ by an analytic continuation argument.

Thus the Hilbert matrix H_k has the same spectrum as the multiplication operator $\pi \operatorname{sech}(\pi\lambda^{1/2})$ on $L^2(d\rho_k)$, and we have our

THEOREM 5.

(i) For all real $k \neq 1, 2, \dots$, H_k has continuous spectra of multiplicity one on $[0, \pi]$;

(ii) If $k \leq 1/2$, H_k has no point spectrum;

(iii) If $k > 1/2$, let p and q be the largest non-negative integers such that $2p < k - 1/2$ and $2q < k - 3/2$ respectively. Then $\pi \operatorname{csc} \pi k$ and $-\pi \operatorname{csc} \pi k$ are eigenvalues of H_k of multiplicities $p+1$ and $q+1$ respectively. H_k has no other point spectrum.

PROOF. The closure of the range of $\pi \operatorname{sech}(\pi\lambda^{1/2})$, $0 \leq \lambda < \infty$ is $[0, \pi]$ so (i) is proved. (iii) follows from an examination of $\pi \operatorname{sech}(\pi\lambda^{1/2})$, $\lambda^{1/2} = i(k - 1/2 - n)$, $n = 0, 1, \dots, N_k$, $N_k < k - 1/2$.

The eigenvalues and corresponding eigenvectors in (iii) were exhibited by Hill [2]. Theorem 5 provides a complete determination of the spectrum of H_k and thus solves a problem posed by Magnus in [7].

REFERENCES

1. A. Erdélyi, *Higher transcendental functions*, I, McGraw-Hill, 1953.
2. C. K. Hill, *The Hilbert bound of a certain doubly-infinite matrix*, J. London Math. Soc. vol. 32 (1957) pp. 7-17.
3. A. E. Ingham, *A note on Hilbert's inequality*, J. London Math. Soc. vol. 11 (1936) pp. 237-240.
4. K. Kodaira, *The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices*, Amer. J. Math. vol. 71 (1949) pp. 921-945.
5. N. N. Lebedev, *Some singular integral equations connected with integral representations of mathematical physics*, C. R. (Doklady) Acad. Sci. URSS. vol. 65 (1949) pp. 621-624.
6. ———, *The analogue of Parseval's theorem for a certain integral transform*, C. R. (Doklady) Acad. Sci. URSS. vol. 68 (1949) pp. 653-656.
7. W. Magnus, *Ueber einige beschränkte Matrizen*, Archiv der Mathematik, vol. 2 (1949-1950) pp. 405-412.
8. ———, *On the spectrum of Hilbert's matrix*, Amer. J. Math. vol. 72 (1950) pp. 699-704.
9. M. Rosenblum, *On the Hilbert matrix I*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 137-140.
10. H. Shanker, *An integral equation for Whittaker's confluent hypergeometric function*, Proc. Cambridge Philos. Soc. vol. 45 (1949) pp. 482-483.
11. E. C. Titchmarsh, *Eigenfunction expansions associated with second order differential equations*, Oxford, 1946.

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