ON COSET REPRESENTATIVES IN GROUPS

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1. In the following we denote by G some fixed group with the subgroups H and K. In a right coset expansion of G with respect to H one may possibly select the representatives for the cosets such that they may also be used for representatives in a left coset expansion with respect to K

$$(1.1) \qquad G = \sum gH = \sum Kg.$$

We then say that there exist *common representatives* for the two expansions. The case $H = K$ is of special interest. Then

$$(1.2) \qquad G = \sum gH = \sum Hg$$

and G has *common left and right representatives with respect to H*. It has been shown by Miller [1] and Chapman [2] that expansions (1.2) exist when H is finite. According to Scorza [3] expansions (1.1) exist when H and K are finite with the same order. In the present paper we shall point out some general cases in which such expansions must exist for infinite subgroups H and K.

2. Let

$$(2.1) \qquad G = \sum KdH$$

be the double coset expansion of G with respect to H and K. It is readily seen that an expansion (1.1) can only exist when there is such an expansion for each of the double cosets in (2.1). A coset expansion of KdH will have the form

$$(2.2) \qquad KdH = \sum Kdh = \sum kdH, \ h \in H, \ k \in K.$$

Here any two cosets Kdh and kdH have a nonvoid intersection since the element kdh belongs to both. When there exist common representatives for the two expansions (2.2) of KdH the cardinal number of cosets of H and K is the same. Conversely, if this is the case there is a one-to-one correspondence

$$H_i \rightleftarrows K_i$$

between the two families of cosets. For each i one can select an arbitrary

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$$g_i \in H_i \cap K_i$$

and these elements then form a system of common representatives in (2.2).

In (2.2) two elements h and h_1 define the same K-coset

$$Kdh = Kdh_1$$

if and only if

$$h_1 h^{-1} \in H \cap d^{-1} K d.$$

Thus the distinct K-cosets in KdH correspond in a one-to-one manner to the cosets in the expansion

$$(2.3) \qquad H = \sum D_H \cdot h_i, \qquad D_H = H \cap d^{-1} K d.$$

Analogously the distinct H-cosets in KdH correspond in a one-to-one manner to the cosets in the expansion

$$(2.4) \qquad K = \sum k_i D_K, \qquad D_K = K \cap d H d^{-1}.$$

This leads immediately to:

THEOREM 2.1. *A necessary and sufficient condition that the group G have common representatives for expansion with respect to H and K is that for all $d \in G$ the two indices*

$$(2.5) \qquad [H/D_H] = [K/D_K]$$

have the same cardinal number. Here the groups D_H and D_K are defined in (2.3) and (2.4).

For $H = K$ the condition for common left and right representatives for a group H becomes

$$(2.6) \qquad [H/H \cap d^{-1} H d] = [H/H \cap d H d^{-1}].$$

3. The groups D_H and D_K are conjugate

$$(3.1) \qquad d D_H d^{-1} = D_K.$$

Thus we conclude:

THEOREM 3.1. *The group G has common representatives for the subgroups H and K when these groups have the property: If H_1 and K_1 are conjugate subgroups in G such that*

$$H \supset H_1, \qquad K \supset K_1$$

then

$$(3.2) \qquad [H/H_1] = [K/K_1].$$

The following criterion depends only on H and K and is independent of G:

THEOREM 3.2. *There exist common representatives for H and K if (3.2) holds for every pair of isomorphic subgroups H_1 and K_1 of H and K respectively.*

When applied to $H = K$ we obtain:

THEOREM 3.3. *G has common left and right coset representatives for H if any two conjugate subgroups of G contained in H have the same index in H; thus in particular if H has the property that any two isomorphic subgroups in it always have the same index.*

These results evidently include the previously mentioned results by Miller, Chapman and Scorza for the case where H and K are finite.

4. Let $d \in G$. A subgroup A is *d-invariant* if

$$dAd^{-1} = A.$$

Every subgroup H of G contains a *maximal d-invariant subgroup $I_d(H)$* given by the formula

(4.1) $$I_d(H) = \wedge H_i, \qquad i = 0, \pm 1, \pm 2, \cdots$$

as the intersection of the conjugates

(4.2) $$H_i = d^i H d^{-i}.$$

The formula (4.1) shows that

(4.3) $$I_d(H) = I_{d^{-1}}(H).$$

All H_i in (4.2) have the same maximal d-invariant subgroup $I_d(H)$. For two arbitrary subgroups A and B

(4.4) $$I_d(A \cap B) = I_d(A) \cap I_d(B).$$

From the index relation

$$[A \cap B/I_d(A \cap B)]$$
$$= [A \cap B/I_d(A) \cap B] \cdot [I_d(A) \cap B/I_d(A) \cap I_d(B)]$$

one concludes that

(4.5) $$[A \cap B/I_d(A \cap B)] \leqq [A/I_d(A)] \cdot [B/I_d(B)].$$

Let us return to the condition (2.5) for the existence of common representatives. According to (3.1) it may be written

(4.6) $$[H/H \cap d^{-1}Kd] = [d^{-1}Kd/H \cap d^{-1}Kd].$$

We denote the maximal d-invariant subgroups of H and K respectively by

(4.7) $$I_d(H), \quad I_d(K).$$

From (4.4) and the preceding remarks we obtain

$$I_d(H \cap K) = I_d(H) \cap I_d(K) = I_d(H) \cap I_d(d^{-1}Kd) = I_d(H \cap d^{-1}Kd).$$

Thus a consequence of (4.6) is

(4.8) $$[H/I_d(H \cap K)] = [d^{-1}Kd/I_d(H \cap K)]$$

and so we have:

THEOREM 4.1. *A necessary condition for the existence of common representatives for H and K is that for any $d \in G$*

(4.9) $$[H/I_d(H \cap K)] = [K/I_d(H \cap K)].$$

Assume that the indices

(4.10) $$[H/I_d(H)], \quad [K/I_d(K)]$$

are finite. It follows from (4.5) that also the indices

$$[H \cap K/I_d(H \cap K)], \quad [H \cap d^{-1}Kd/I_d(H \cap K)]$$

are finite. Thus one can conclude reversely from (4.9) to (2.5) through (4.8) and (4.6). Therefore in this case the condition (4.9) is both necessary and sufficient. But from the index relation

$$[H/I_d(H \cap K)] = [H/H \cap K] \cdot [H \cap K/I_d(H \cap K)]$$

and analogously for K we conclude further:

THEOREM 4.2. *Let H and K be subgroups such that the indices (4.10) are finite for every $d \in G$. Then there exist common representatives for H and K if and only if*

(4.11) $$[H/H \cap K] = [K/H \cap K].$$

It is well known that when H and K are subgroups of G of finite index then each of them contains a normal subgroup of G also of finite index. We conclude that in this case the indices (4.10) are finite. But then the condition (4.11) can be replaced by

$$[G/H] = [G/K]$$

and so we may state:

THEOREM 4.3. *When H and K are subgroups of finite index in G then they have common representatives if and only if they have the same index in G.*

For $H = K$ the condition for common right and left representatives in Theorem 4.2 becomes simply that the index

$$(4.12) \qquad\qquad [H/I_d(H)]$$

shall be finite. This is always fulfilled when H has finite index in G.

5. We shall investigate the case where the index (4.12) is infinite. For $d \in G$ let H and B be subgroups such that

$$(5.1) \qquad\qquad H \supset B \cup dBd^{-1}.$$

Then

$$[H/B] \geqq [dBd^{-1}/B \cap dBd^{-1}]$$

and consequently

$$(5.2) \qquad\qquad [H/B] \geqq [B/B \cap d^{-1}Bd].$$

We shall put

$$I_k(B, d) = B \cap d^{-1}Bd \cap \cdots \cap d^{-k}Bd^k.$$

Since

$$dI_kd^{-1} = dBd^{-1} \cap I_{k-1} \subset I_{k-1}$$

the inequality (5.2) may be applied repeatedly so that one obtains

$$[H/B] \geqq [I_{k-1}/I_k].$$

From the index relation

$$[H/I_k] = [H/I_0] \cdot [I_0/I_1] \cdots [I_{k-1}/I_k]$$

we conclude that

$$(5.3) \qquad\qquad [H/I_k] \leqq [H/B]^{k+1}.$$

Let us say that $I_d(H)$ in (4.1) has the *finite intersection property* if it is the intersection of a finite number of the conjugates (4.2). This is the case, in particular, when there is only a finite number of such conjugates. In (5.3) we next make use of the fact that for any infinite cardinal one has

$$\alpha^n = \alpha, \qquad n \text{ finite.}$$

It then follows:

THEOREM 5.1. *Let H and B be subgroups satisfying* (5.1) *and suppose that $I_d(B)$ has the finite intersection property. Then the indices*

$$[H/B], \qquad [H/I_d(B)]$$

are either both finite or both infinite and in the latter case they are equal.

In connection with this theorem let us mention the following analogue for normal subgroups: Let H be a subgroup of G and N the greatest normal subroup of G contained in H; thus N is the intersection of all conjugates of H. If N can be represented as the intersection of a finite number of such conjugates the indices

$$[G/H], \qquad [G/N]$$

are either both finite or both infinite and in the latter case

$$[G/H] = [G/N].$$

We shall apply Theorem 5.1 to

$$B = H \cap d^{-1}Hd$$

so that

$$I_d(B) = I_d(H).$$

It follows:

THEOREM 5.2. *Let $I_d(H)$ have the finite intersection property. Then*

(5.4) $$[H/I_d(H)] = [H/H \cap d^{-1}Hd]$$

when the first of these indices is infinite.

From (5.4) we obtain (2.6) by means of the observation (4.3) and so we may state:

THEOREM 5.3. *The group G has common left and right coset representatives for the group H when for every $d \in G$ the maximal d-invariant subgroup $I_d(H)$ of H has the finite intersection property.*

The condition is satisfied when H has a finite number of conjugates by transformation with the powers of any element $d \in G$. Theorem 5.3 then gives a result due to Shü [4].

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