THE INTERIOR POINTS OF THE PRODUCT OF TWO SUBSETS OF A LOCALLY COMPACT GROUP

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1. **Introduction.** Recently, B. J. Pettis conjectured that under certain hypotheses (see Theorem 1 below) the product of two subsets of a locally compact topological group should have a nonvoid interior. The purpose of this paper is to prove this statement. We then use this result to prove, among other things, a theorem of Simon's [**1**], which he established by a different method.

The conjecture is an analog of earlier work by E. J. McShane [**2**] and B. J. Pettis [**3**; **4**], namely:

THEOREM (MCSHANE). *Let G be a topological group and let A, $B\subset G$ be sets of second category with one of them satisfying the condition of Baire. Then the interior of AB is not empty.*

Our theorem reads:

THEOREM 1. *Let G be a locally compact topological group with completed Haar measure μ and outer measure μ^*. Let A, $B\subset G$ be sets such that $\mu(A)>0$ and $\mu^*(B)>0$. Then the interior of BA (also AB) is nonvacuous.*

In order to show this, we prove the following lemma:

LEMMA 1.1. *Let G be a locally compact group with completed Haar measure μ and outer measure μ^*. Let A, $B\subset G$ be sets such that A has finite measure and B is contained in a set of finite measure. If the function f is defined by $f(x)=\mu^*(xA\cap B)$, then f is continuous and $\int_G f(x)d\mu(x) =\mu(A^{-1})\mu^*(B)$.*

As an application of Theorem 1, we prove:

THEOREM 3 (SIMON [**1**]). *In a compact group, every subsemigroup which contains a set of positive measure is an open and closed subgroup; and therefore is itself measurable.*

2. **Proof of Theorem 1.** We first prove Lemma 1.1 which was stated in the introduction.

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Proof of Lemma 1.1. To show f is continuous, let $\epsilon>0$ be given. We can find a neighborhood U of the identity with the property that $\mu(xA\Delta xuA)=\mu(A\Delta uA)<\epsilon$ for all $u\in U$ and $x\in G$. Now $(xA\cap B)\cdot\Delta(xuA\cap B)=(xA\Delta xuA)\cap B$; hence $\mu^*((xA\cap B)\Delta(xuA\cap B))<\epsilon$, if $u\in U$. Since xA and xuA are measurable, we have

$$\mu^*(xA \cap B) = \mu^*((xA \cap B) \cap xuA) + \mu^*((xA \cap B)\backslash xuA),$$

and

$$\mu^*(xuA \cap B) = \mu^*((xuA \cap B) \cap xA) + \mu^*((xuA \cap B)\backslash xA).$$

Applying these formulas to the definition of f, it follows that $|f(x) - f(xu)| = |\mu^*((xA \cap B)\backslash xuA) - \mu^*((xuA \cap B)\backslash xA)| \leqq \mu^*((xA\cap B)\Delta(xuA\cap B))<\epsilon$ for all $u\in U$ and $x\in G$; and f is continuous.

Now let $\mu(A^{-1})=\alpha<\infty$ and $\mu^*(B)=\beta<\infty$, and let the sequence of sets $\{B_n\}$ be chosen so that $B\subset B_{n+1}\subset B_n$; B_n is measurable; and $\mu(B_n)\leqq\beta+1/n$, for $n=1, 2, \cdots$. Let $f_n(x)=\mu(xA\cap B_n)$. Then $f_n\geqq f_{n+1}\geqq 0$; f_n is continuous; and $\int f_n(x)d\mu(x)=\int\mu(xA\cap B_n)d\mu(x)=\int(\int\chi_{xA}(y)\chi_{B_n}(y)dy)dx=\int\chi_{B_n}(y)(\int\chi_{yA^{-1}}(x)dx)dy=\mu(A^{-1})\mu(B_n)$ for $n=1, 2, \cdots$. Now $\alpha\beta=\mu(A^{-1})\mu^*(B)\leqq\mu(A^{-1})\mu(B_n)\leqq\alpha(\beta+1/n)$, for all n. Therefore, if we set $\bar{f}=\inf_n f_n$, then $\bar{f}$ is measurable and $\int_G\bar{f}(x)d\mu(x)=\alpha\beta$. It remains only to show $f(x)=\bar{f}(x)$, i.e. that

$$\mu^*(xA \cap B) = \inf_n \mu(xA \cap B_n).$$

We know that

$$\mu^*(B) = \mu^*(xA \cap B) + \mu^*(B\backslash xA);$$

$$\mu(B_n) = \mu(xA \cap B_n) + \mu(B_n\backslash xA);$$

and clearly,

$$\mu(xA \cap B_n) \geqq \mu^*(xA \cap B) \text{ and } \mu(B_n\backslash xA) \geqq \mu^*(B\backslash xA).$$

Therefore, since $\mu^*(B)=\inf_n \mu(B_n)$, it must be that $\mu^*(xA\cap B)=\inf_n \mu(xA\cap B_n)$ and the lemma is proved.

Remarks. (1) Note that we have actually proved f is uniformly continuous. (2) Easy examples, such as a strip in the plane, show that the assertion is false when the measure of A is infinite.

Proof of Theorem 1. There is no loss of generality in assuming A and B have finite outer measure. Define $f(x)=\mu^*(xA^{-1}\cap B)$, and let $V=\{x:f(x)>0\}$; by Lemma 1.1, V is a nonempty open set. If $x\in V$, then $xA^{-1}\cap B$ cannot be empty, i.e., there are elements a, b

of A and B, respectively, such that $xa^{-1}=b$; $x=ba$. This argument shows that $V\subset BA$ and concludes the proof.

We remark here that, in general, the above theorems are not true if neither A nor B is required to be measurable.

3. **Applications.** Obviously, Theorem 1 provides us with considerable information concerning semigroups in a topological group. We formulate, here, some of the applications. Unless otherwise stated, G is assumed to be a locally compact topological group with invariant Haar measure μ.

THEOREM 2. *Let S be a subsemigroup of G which contains a subset of positive measure. Then the interior of S is not void. In particular,* (1) *if S is a group then S is open and closed, and* (2) *if G is n-dimensional Euclidean space E^n, then S contains an open sector.*

PROOF. The first assertion is a consequence of Theorem 1 and the definition of a semigroup. If a group contains a nonempty open set, then it is open and closed which implies (1), while (2) follows from the observation that every open set in E^n generates a semigroup containing a sector.

PROOF OF THEOREM 3. (The statement appears in the introduction.) By Theorem 2, if S is a semigroup which contains a set of positive measure, then S^0 (=the interior of S) is nonvoid. Now S^0 is an open semigroup, since the interior of a semigroup is again a semigroup, and we now employ a result of Fred B. Wright [5] which asserts that an open semigroup of a compact group is a closed subgroup. Thus, the identity e of G is in the interior of S, and it follows that S is open; S is an open and closed subgroup.

We have shown that for "large" A and B it often happens that AB contains an open set. On the other hand *if G is a commutative compact group with an element of infinite order, it will follow from Theorem* 3 *that there are subsets S and T such that $S\cup T=G$ and neither S^2 nor T^2 contains a nonvoid open set.*

To prove this last statement, we will need the following purely algebraic lemma:

LEMMA 4.1. *Let H be an Abelian group and let $h\in H$. If $S\subset H$ is a semigroup containing h but not containing $-h$, and if S is maximal with respect to this property, then $S\cup -S=H$.*

PROOF. Suppose, to the contrary, that there is an element $t\in H$ such that $t\notin S\cup -S$. Obviously $-t\notin S\cup -S$, so by the maximality of S, it follows that there is some positive integer n and $s_1\in S$ with

$-h=s_1+n(-t)$. Therefore, we may now pick n' to be the smallest positive integer with the property $n't\in S$. Similarly, there is a positive integer m and an $s_2\in S$ such that $-h=s_2+mt$. There exist positive integers p, r with $0\leqq r<n'$ and $m=pn'-r$; but $rt=-mt+pn't=s_2+h+pn't\in S$; hence $r=0$. This gives $-h=s_2+pn't\in S$, a contradiction, and the proof is complete.

As for the statement preceding Lemma 4.1, assume that G is a compact Abelian group having an element of infinite order. By Zorn's Lemma, the S of Lemma 4.1 exists. Moreover, $S^2\subset S$ cannot contain an open set or S would be a group, by Theorem 3; the same statement holds for $-S$; and $S\cup -S=G$.

We now make the further claim that S is not measurable. For, if it were, then its measure must be positive and its interior would be nonempty. In particular, we have proved the following:

THEOREM 4. *In a compact Abelian group, every maximal semigroup which is not a group is not Haar measurable.*

Of course, such semigroups may not exist, but if the group has one element of infinite order, then it can be shown that there is a maximal semigroup which is not a group.

Our final theorem shows that the term "semigroup" is not a misnomer. Roughly, it says that if a semigroup is "more than one-half of a group," then it is indeed a group.

THEOREM 5. *Let S and T be measurable subsets of G such that* (1) *S is a semigroup,* (2) *T is a group, and* (3) $\mu(S\Delta T)<\mu(T)/2$. Then $S=T$.

We first prove the following lemma.

LEMMA 5.1. *If A and B are measurable subsets of G such that* (1) *A is a semigroup,* (2) *B is a group,* (3) *$B\subset A$, and* (4) $\mu(A\backslash B)<\mu(B)$. *Then $A=B$.*

PROOF. By way of contradiction, suppose that $x\in A\backslash B$. Then, since A is a semigroup and B is a group, we have $(xB)\subset A\backslash B$. But $\mu(xB)=\mu(B)>\mu(A\backslash B)$, and the lemma follows.

In the theorem, first consider the case where $\mu(T)=\infty$. If A is the set of $s\in S\cap T$ such that $s^{-1}\notin S\cap T$, then $A^{-1}\subset T\backslash S$. Therefore $\mu(S\cap S^{-1}\cap T)=\infty$ and $\mu(T\backslash(S\cap S^{-1}))<\infty$. The lemma then implies that $T=S\cap S^{-1}\cap T$. We now have a group T in S satisfying the hypothesis of the lemma; that is $T=S$.

In case $\mu(T)<\infty$, then T is compact; and $S\cap T$ is a semigroup of positive measure in T. So by Theorem 3 and the lemma, $S\cap T=T$; $S=T$.

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SIMPLE NODAL NONCOMMUTATIVE JORDAN ALGEBRAS

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1. Introduction. Nodal algebras were defined by R. D. Schafer [**4**] and have also been studied by the author [**2**; **3**]. A noncommutative Jordan algebra is an algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ satisfying (1) the flexible law $(xy)x=x(yx)$ and (2) the condition that $\mathfrak{A}^+$ is a Jordan algebra. That is, $\mathfrak{A}^+$ satisfies the identity $(x^2\cdot y)\cdot x=x^2\cdot(y\cdot x)$ where we have used the dot to indicate the product of $\mathfrak{A}^+$. The algebra $\mathfrak{A}^+$ is defined to be the same vector space as $\mathfrak{A}$ but with product $x\cdot y=(xy+yx)/2$ where xy and yx are products in $\mathfrak{A}$. Then $\mathfrak{A}$ is called nodal if it is finite dimensional, if $\mathfrak{A}$ has identity element 1, if $\mathfrak{A}$ can be written as a vector space direct sum $\mathfrak{A}=\mathfrak{F}1+\mathfrak{N}$ where $\mathfrak{N}$ is the subspace of nilpotent elements of $\mathfrak{A}$, and if $\mathfrak{N}$ is not a subalgebra of $\mathfrak{A}$.

Every known nodal algebra $\mathfrak{A}$ has the property that $\mathfrak{A}^+$ is an associative algebra. The flexible algebras with $\mathfrak{A}^+$ associative have been described in [**3**]. In this paper we shall prove the following theorem.

Theorem 1. *Let $\mathfrak{A}$ be a simple nodal noncommutative Jordan algebra of characteristic $\neq 2$. Then $\mathfrak{A}^+$ is associative.*

Define $\mathfrak{B}$ to be the subspace of $\mathfrak{A}$ generated by all the associators in $\mathfrak{N}^+$. That is, $\mathfrak{B}$ is generated by elements of the form $(x\cdot y)\cdot z-x\cdot(y\cdot z)$ with x, y, z in $\mathfrak{N}$. The proof of the theorem will be made by showing that the ideal $\mathfrak{C}$ of $\mathfrak{A}$ generated by $\mathfrak{B}$ is not all of $\mathfrak{A}$ and since $\mathfrak{A}$ is simple it will follow that $\mathfrak{C}=0$ and $\mathfrak{B}=0$. This is the desired result.

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