AN EXTENSION OF A THEOREM OF MANDELBROJT[1]

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1. **Introduction.** In a recent paper [**3**] S. Mandelbrojt proved several interesting theorems concerning Fourier transforms and analytic functions. One of these results can be formulated as follows:

(1.1) *Suppose* $F \in L_\infty$, $a \geqq 0$, *and that* k *is never zero outside the closed interval* $[-a, \infty]$ *where* k *is the Fourier transform of a function* K *in* L_1 *such that* $K * F \equiv 0$. *Then there exists a function* F_0 *analytic in the right half plane with the properties*

$$\text{(i)} \qquad | F_0(x + iy) | \leqq \|F\|_\infty e^{ax}, \qquad x > 0,$$

$$\text{(ii)} \qquad \lim_{x \to +0} \int_{-N}^{N} | F_0(x + iy) - F(y) | \, dy = 0.$$

It is not difficult to show that the conclusion is satisfied if we assume only that there exists for each t in $(-\infty, -a)$ a function K in L_1 (depending upon t) such that $K * F \equiv 0$ and $k(t) \neq 0$. Our principal aim in this paper is to extend this latter improved version to n-dimensions. That such an extension exists follows almost immediately from a theorem of Y. Fourés and I. E. Segal [**1**] concerning causal operators and analytic functions (once it is established that a certain bounded operator T determined by F is causal). On the other hand as indicated by these authors some of the results in [**1**], specifically those pertaining to domains of dependence, admit improvement when treated from the point of view of Banach algebras; as one is led naturally to making these improvements in the course of showing that T is causal we shall begin our discussion at this point.

2. **Domains of dependence.** Throughout this part $\mathcal{G}$ will denote an arbitrary locally compact abelian group. As is well known, the Plancherel transform $U(U: L_2(\mathcal{G}) \to L_2(\widehat{\mathcal{G}}))$ establishes a one to one correspondence between bounded operators T on $L_2(\mathcal{G})$ that commute with translations and bounded measurable functions F on the dual group $\widehat{\mathcal{G}}$. More precisely $T \leftrightarrow F$ if and only if $UTU^{-1} = M_F$ where M_F denotes the operation of multiplication by F on $L_2(\widehat{\mathcal{G}})$. We recall that the spectrum or spectral set $\Lambda(F)$ of F is defined as the set of all x in $\mathcal{G}$ such that $k(x) = 0$ whenever k is the (inverse) Fourier transform

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of a function K in $L_1(\widehat{\mathcal{G}})$ such that $K * F \equiv 0$. The set of all such K forms a closed ideal I in $L_1(\widehat{\mathcal{G}})$. A theorem of Segal's [2] asserts that

(2.1) *A sufficient condition that $K \in I$ is that its Fourier transform k vanishes outside a compact subset of the complement $\Lambda(F)'$ of $\Lambda(F)$.*

Definition 2.1. A bounded operator T is said to be *dependent* upon a subset E of $\mathcal{G}$ if Tg vanishes outside $N+E$ whenever N is compact, $g \in L_2$ and g vanishes outside N.

Definition 2.2. A bounded operator T has a *domain* of *dependence* if there exists a closed subset E of $\mathcal{G}$ such that T is dependent upon E and is not dependent upon any proper closed subset of E. E is called a domain of dependence of T.

Theorem 2.1. *Suppose T is a bounded operator on $L_2(\mathcal{G})$ that commutes with translations. Then T has a unique domain of dependence E, and if $UTU^{-1} = M_F$, E is precisely the spectrum of F.*[2]

Lemma 2.1. *If T is dependent upon a closed set E then $\Lambda(F) \subset E$.*

Let $e \in E'$ and choose a compact neighborhood N of 0 such that $E+N$ and $e+N$ are disjoint. Suppose g, $h \in L_2$ and vanish outside N, $e+N$. Then $(Tg)h=0$ and taking Fourier transforms we get $(FG) * H \equiv 0$. Let x be a fixed character of $\mathcal{G}$ (i.e. an element of $\widehat{\mathcal{G}}$) and denote the value of x at $t \in \mathcal{G}$ by $(x \cdot t)$. Replacing $g(t)$ by $k_x(t) = (x \cdot t)[g(t)]^-$ we see that $K(u) = \int (u \cdot t) k_x(t) dt = [G(x-u)]^-$. Hence $\int F(u-v)[G[x-(u-v)]]^- H(v)dv = 0$ for all u and putting $u = x$ we get $\int F(x-v)[G(v)]^- H(v) dv = 0$. As x was arbitrary in $\mathcal{G}$ it follows that $F * \overline{G}H \equiv 0$. Now by choosing g and h suitably (subject to the above restrictions) we can insure that $g^* * h(e) \neq 0$. Since $g^* * h$ is the (inverse) Fourier transform of $\overline{G}H$ it follows that $e \in \Lambda(F)'$, and consequently $\Lambda(F) \subset E$.

Proof of the theorem. Since $\Lambda(F)$ is closed and in view of the result just established it suffices to show that T is dependent upon $E = \Lambda(F)$. Suppose then that $g \in L_2$ and vanishes outside a compact subset N. To show that Tg vanishes outside $E+N$ it suffices to show that $(Tg, h) = 0$ for every h in L_2, vanishing outside a compact subset C of $(E+N)'$. By Parseval's formula, it suffices to show that $\int FG\overline{H} = 0$. Now $\int FG\overline{H} = \int F[\overline{G}H]^- = F * (\overline{G}H)^*(o)$ and it is therefore sufficient to show that $F * (\overline{G}H)^* \equiv 0$. The Fourier transform of $(\overline{G}H)^*$ is $[g^* * h]^-$ and as g vanishes outside N, g^* vanishes outside $-N$. Thus $[g^* * h]^-$ vanishes outside $C-N$. Furthermore $C-N$ is compact and disjoint from E. Consequently (2.1) applies and we see that $F * (\overline{G}H)^* \equiv 0$.

[2] H. Helson obtained a similar characterization of the spectrum in an unpublished part of his thesis—Harvard, 1950.

REMARK. In the case of a real (finite dimensional) vector group the domain of dependence for T in the sense of Fourés-Segal is simply the closed convex set generated by $\Lambda(F)$.

3. **Mandelbrojt's theorem.** Throughout this part $\mathcal{G}$ will denote n-dimensional real Euclidean space regarded as a vector group. The dual of a cone C in $\mathcal{G}$ is the cone $\hat{C}$ in the dual group $\widehat{\mathcal{G}}$ consisting of all x such that $(x \cdot t) \geqq 0$ for all t in C. The tube Γ over $\hat{C}$ is the set of all complex vectors $x+iy$, $x \in \hat{C}$, $y \in \widehat{\mathcal{G}}$. Putting v for the vertex of C we define the spine of Γ to be the subset of all $v+iy$, $y \in \widehat{\mathcal{G}}$. By means of an obvious correspondence we can identify functions on $\widehat{\mathcal{G}}$ with functions on the spine of Γ. A function F_0 defined on the interior Γ^0 of Γ is said to extend a function F on the spine, or to have boundary values on the spine if any sequence $x_n \to v$ with x_n in $\hat{C}^0$ has a subsequence x_m such that $F_0(x_m+iy) \to F(v+iy)$ a.e. relative to Lebesque measure.

A bounded operator T on $L_2(\mathcal{G})$ is said to be *causal* with respect to C if Tg vanishes outside $a+C$ whenever g is in L_2 and vanishes outside $a+C$, a being arbitrary in $\mathcal{G}$. We shall use the following reformulation of the basic result concerning bounded causal operators given in [1].

(3.1) *Suppose C is a closed convex cone with vertex at* 0 *and nonempty interior. Let T be a bounded operator on $L_2(\mathcal{G})$ that commutes with translations, and suppose F is the unique (modulo null functions) bounded measurable function on $\widehat{\mathcal{G}}$ such that $UTU^{-1} = M_F$ where U is the Plancherel transform,*

$$U: L_2(\mathcal{G}) \to L_2(\widehat{\mathcal{G}})$$

and M_F is the operation of multiplication by F on $L_2(\widehat{\mathcal{G}})$. Then T is causal with respect to C if and only if there exists a function F_0 analytic on the interior of the tube Γ over the dual of C that extends F and satisfies the additional conditions,

$$\text{(i)} \qquad | F_0(z) | \leqq \|F\|_\infty, \qquad z \in \Gamma^0$$

$$\text{(ii)} \qquad \lim_{x \to 0} \int_D | F_0(x+iy) - F(y) |^2 dy = 0$$

where $x \to 0$ in $\hat{C}^0$ and D is an arbitrary compact subset of the spine of Γ.

Our extension of Mandelbrojt's theorem reads as follows.

THEOREM 3.1. *Suppose $F \in L_\infty(\widehat{\mathcal{G}})$ and that C is a closed convex cone in $\mathcal{G}$ with vertex at* 0 *and nonempty interior. Let a be a fixed element in C. Suppose further that there exists for each t outside $C-a$ a function K in $L_1(\widehat{\mathcal{G}})$ such that $K * F \equiv 0$ and $k(t) \neq 0$ where k is the (inverse) Fourier* *transform of K. Then there exists a function F_0 analytic on the interior of the tube Γ over the dual of C extending F and having the additional properties,*

$$\text{(i)} \qquad |F_0(x+iy)| \leqq \|F\|_\infty e^{a\cdot x}$$

for all x in $\hat{C}^0$ and y in $\widehat{\mathcal{G}}$.

$$\text{(ii)} \qquad \lim_{x\to 0} \int_D |F_0(x+iy) - F(y)|^2 dy = 0$$

where $x \to 0$ in $\hat{C}^0$ and D is an arbitrary compact subset of the spine of Γ.

It is easy to see that the theorem follows, by translation, from the case $a=0$. The details of this reduction are given in the following

LEMMA 3.1. *If the theorem is true for $a=0$ it is true in general.*

Suppose $F \in L_\infty(\widehat{\mathcal{G}})$ and that the additional hypotheses of the theorem are satisfied. Put $H(x) = e^{-i(a\cdot x)}F(x)$ and suppose $t \in C'$. Then $t-a \in (C-a)'$ and there exists K in $L_1(\widehat{\mathcal{G}})$ such that $K * F \equiv 0$ and $k(t-a) \neq 0$ where $k(u) = (2\pi)^{-n/2}\int e^{i(u\cdot x)}K(x)dx$. Putting $L(x) = e^{-i(a\cdot x)}K(x)$ it follows that $1(t) = k(t-a) \neq 0$ and that $L * H(x) = e^{-(a\cdot x)}K * F(x) = 0$. Assume the theorem is true for the case $a=0$, and let H_0 be the extension of H. Define F_0 by

$$F_0(x+iy) = e^{a\cdot(x+iy)}H_0(x+iy)$$

for $x \in \hat{C}^0$ and $y \in \widehat{\mathcal{G}}$. Then $|F_0(x+iy)| \leqq e^{a\cdot x}\|H\|_\infty = e^{a\cdot x}\|F\|_\infty$, and if D is a compact subset of the spine of Γ we have,

$$\begin{aligned}\int_D |F_0(x+iy) - F(y)|^2 dy &= \int_D |e^{a\cdot(x+iy)}H_0(x+iy) - e^{ia\cdot y}H(y)|^2 dy \\ &\leqq \int_D |e^{a\cdot x}H_0(x+iy) - H(iy)|^2 dy.\end{aligned}$$

Now $\int_D |e^{a\cdot x}H_0(x+iy) - H_0(x+iy)|^2 dy = (e^{a\cdot x}-1)^2\int_D |H_0(x+iy)|^2 dy \to 0$ as $x \to 0$ in $\hat{C}^0$, in view of (ii), and the fact that $(e^{a\cdot x}-1)^2 \to 0$. Combining these estimates we see that $\int_D |F_0(x+iy) - F(y)|^2 dy \to 0$ as $x \to 0$ through values in $\hat{C}^0$.

PROOF OF THE THEOREM. By the preceding lemma we can assume $a=0$. Now let T be the bounded operator on $L_2(\mathcal{G})$ given by the equation $T = U^{-1}M_F U$. It is apparent from (3.1) that it suffices to show that T is causal with respect to the cone C. Since T commutes with translations we need only show that Tg vanishes outside C whenever g is in L_2 and vanishes outside C. Furthermore as T is bounded it suffices, by continuity, to consider the case that g vanishes outside a compact subset N of C. Now our assumptions clearly imply that the spectrum E of F is contained in C, and, by Theorem 2.1, T is therefore dependent upon E. Hence Tg vanishes outside $E+N$. Finally since C is closed under addition, $E+N$ is contained in C which concludes the proof.

References

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