AN EXTENSION OF A THEOREM OF MANDELBROJT

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1. Introduction. In a recent paper [3] S. Mandelbrojt proved several interesting theorems concerning Fourier transforms and analytic functions. One of these results can be formulated as follows:

(1.1) Suppose \( F \in L_a^r \), \( a \geq 0 \), and that \( k \) is never zero outside the closed interval \( [-a, \infty) \) where \( k \) is the Fourier transform of a function \( K \) in \( L_1 \) such that \( K \ast F \equiv 0 \). Then there exists a function \( F_0 \) analytic in the right half plane with the properties

\[
|F_0(x + iy)| \leq \|F\|e^{\alpha x}, \quad x > 0, \\
\lim_{x \to +0} \int_{-N}^{N} |F_0(x + iy) - F(y)| \, dy = 0.
\]

It is not difficult to show that the conclusion is satisfied if we assume only that there exists for each \( t \in (-\infty, -a) \) a function \( K \) in \( L_1 \) (depending upon \( t \)) such that \( K \ast F \equiv 0 \) and \( k(t) \neq 0 \). Our principal aim in this paper is to extend this latter improved version to \( n \)-dimensions. That such an extension exists follows almost immediately from a theorem of Y. Foures and I. E. Segal [1] concerning causal operators and analytic functions (once it is established that a certain bounded operator \( T \) determined by \( F \) is causal). On the other hand as indicated by these authors some of the results in [1], specifically those pertaining to domains of dependence, admit improvement when treated from the point of view of Banach algebras; as one is led naturally to making these improvements in the course of showing that \( T \) is causal we shall begin our discussion at this point.

2. Domains of dependence. Throughout this part \( G \) will denote an arbitrary locally compact abelian group. As is well known, the Plancherel transform \( U(U: L_2(G) \rightarrow L_2(\hat{G})) \) establishes a one to one correspondence between bounded operators \( T \) on \( L_2(G) \) that commute with translations and bounded measurable functions \( F \) on the dual group \( \hat{G} \). More precisely \( T \leftrightarrow F \) if and only if \( UTU^{-1} = M_F \) where \( M_F \) denotes the operation of multiplication by \( F \) on \( L_2(\hat{G}) \). We recall that the spectrum or spectral set \( \Lambda(F) \) of \( F \) is defined as the set of all \( x \) in \( G \) such that \( k(x) = 0 \) whenever \( k \) is the (inverse) Fourier transform

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of a function $K$ in $L_1(\mathcal{G})$ such that $K \ast F = 0$. The set of all such $K$ forms a closed ideal $I$ in $L_1(\mathcal{G})$. A theorem of Segal's [2] asserts that

(2.1) A sufficient condition that $K \in I$ is that its Fourier transform $k$ vanishes outside a compact subset of the complement $\Lambda(F)'$ of $\Lambda(F)$.

**Definition 2.1.** A bounded operator $T$ is said to be dependent upon a subset $E$ of $\mathcal{G}$ if $Tg$ vanishes outside $N + E$ whenever $N$ is compact, $g \in L_2$ and $g$ vanishes outside $N$.

**Definition 2.2.** A bounded operator $T$ has a domain of dependence if there exists a closed subset $E$ of $\mathcal{G}$ such that $T$ is dependent upon $E$ and is not dependent upon any proper closed subset of $E$. $E$ is called a domain of dependence of $T$.

**Theorem 2.1.** Suppose $T$ is a bounded operator on $L_2(\mathcal{G})$ that commutes with translations. Then $T$ has a unique domain of dependence, and if $UTU^{-1} = M_F$, $E$ is precisely the spectrum of $F$.  

**Lemma 2.1.** If $T$ is dependent upon a closed set $E$ then $\Lambda(F) \subseteq E$.

Let $e \in E'$ and choose a compact neighborhood $N$ of 0 such that $E + N$ and $e + N$ are disjoint. Suppose $g, h \in L_2$ and vanish outside $N$, $e + N$. Then $(Tg)h = 0$ and taking Fourier transforms we get $(FG) \ast H = 0$. Let $x$ be a fixed character of $\mathcal{G}$ (i.e. an element of $\mathcal{G}$) and denote the value of $x$ at $t \in \mathcal{G}$ by $(x \cdot t)$. Replacing $g(t)$ by $k_x(t) = (x \cdot t)[g(t)]^-$ we see that $K(u) = \int (u \cdot t)k_x(t)dt = [G(x - u)]^-$. Hence $\int F(u - v)[G[x - (u - v)]]^-H(v)dv = 0$ for all $u$ and putting $u = x$ we get $\int F(x - v)[G(v)]^-H(v)dv = 0$. As $x$ was arbitrary in $\mathcal{G}$ it follows that $F \ast \overline{GH} = 0$. Now by choosing $g$ and $h$ suitably (subject to the above restrictions) we can insure that $g \ast h(e) \neq 0$. Since $g \ast h$ is the (inverse) Fourier transform of $\overline{GH}$ it follows that $e \in \Lambda(F)'$, and consequently $\Lambda(F) \subseteq E$.

**Proof of the theorem.** Since $\Lambda(F)$ is closed and in view of the result just established it suffices to show that $T$ is dependent upon $E = \Lambda(F)$. Suppose then that $g \in L_2$ and vanishes outside a compact subset $N$. To show that $Tg$ vanishes outside $E + N$ it suffices to show that $(Tg, h) = 0$ for every $h \in L_2$, vanishing outside a compact subset $C$ of $(E + N)'$. By Parseval's formula, it suffices to show that $\int FG\overline{H} = 0$. Now $\int FG\overline{H} = \int F[\overline{GH}]^- = F \ast (\overline{GH})^* (0)$ and it is therefore sufficient to show that $F \ast (\overline{GH})^* \equiv 0$. The Fourier transform of $(\overline{GH})^*$ is $[g \ast h]^-$ and as $g$ vanishes outside $N$, $g$ vanishes outside $-N$. Thus $[g \ast h]^-$ vanishes outside $C - N$. Furthermore $C - N$ is compact and disjoint from $E$. Consequently (2.1) applies and we see that $F \ast (\overline{GH})^* \equiv 0$.

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Remark. In the case of a real (finite dimensional) vector group the domain of dependence for $T$ in the sense of Fourés-Segal is simply the closed convex set generated by $\Lambda(F)$.

3. Mandelbrojt's theorem. Throughout this part $\mathfrak{g}$ will denote $n$-dimensional real Euclidean space regarded as a vector group. The dual of a cone $C$ in $\mathfrak{g}$ is the cone $\hat{C}$ in the dual group $\hat{\mathfrak{g}}$ consisting of all $x$ such that $(x \cdot t) \geq 0$ for all $t$ in $C$. The tube $\Gamma$ over $\hat{C}$ is the set of all complex vectors $x + iy$, $x \in \hat{C}$, $y \in \hat{\mathfrak{g}}$. Putting $\nu$ for the vertex of $C$ we define the spine of $\Gamma$ to be the subset of all $v + iy$, $y \in \hat{\mathfrak{g}}$. By means of an obvious correspondence we can identify functions on $\hat{\mathfrak{g}}$ with functions on the spine of $\Gamma$. A function $F_0$ defined on the interior $\Gamma^0$ of $\Gamma$ is said to extend a function $F$ on the spine, or to have boundary values on the spine if any sequence $x_n \to v$ with $x_n$ in $\hat{C}^0$ has a subsequence $x_m$ such that $F_0(x_m + iy) \to F(v + iy)$ a.e. relative to Lebesgue measure.

A bounded operator $T$ on $L_2(\mathfrak{g})$ is said to be causal with respect to $C$ if $Tg$ vanishes outside $a + C$ whenever $g$ is in $L_2$ and vanishes outside $a + C$, a being arbitrary in $\mathfrak{g}$. We shall use the following reformulation of the basic result concerning bounded causal operators given in [1].

(3.1) Suppose $C$ is a closed convex cone with vertex at 0 and nonempty interior. Let $T$ be a bounded operator on $L_2(\mathfrak{g})$ that commutes with translations, and suppose $F$ is the unique (modulo null functions) bounded measurable function on $\hat{\mathfrak{g}}$ such that $UTU^{-1} = MF$ where $U$ is the Plancherel transform,

$$U: L_2(\mathfrak{g}) \to L_2(\hat{\mathfrak{g}})$$

and $MF$ is the operation of multiplication by $F$ on $L_2(\mathfrak{g})$. Then $T$ is causal with respect to $C$ if and only if there exists a function $F_0$ analytic on the interior of the tube $\Gamma$ over the dual of $C$ that extends $F$ and satisfies the additional conditions,

(i) \[ |F_0(z)| \leq ||F||_\infty, \quad z \in \Gamma^0 \]

(ii) \[ \lim_{z \to 0} \int_D |F_0(x + iy) - F(y)|^2 dy = 0 \]

where $x \to 0$ in $\hat{C}^0$ and $D$ is an arbitrary compact subset of the spine of $\Gamma$.

Our extension of Mandelbrojt's theorem reads as follows.

Theorem 3.1. Suppose $F \in L_\infty(\mathfrak{g})$ and that $C$ is a closed convex cone in $\mathfrak{g}$ with vertex at 0 and nonempty interior. Let $a$ be a fixed element in $C$. Suppose further that there exists for each $t$ outside $C - a$ a function $K$ in $L_1(\mathfrak{g})$ such that $K \ast F \equiv 0$ and $k(t) \neq 0$ where $k$ is the (inverse) Fourier
transform of $K$. Then there exists a function $F_0$ analytic on the interior of the tube $\Gamma$ over the dual of $C$ extending $F$ and having the additional properties,

(i) \[ |F_0(x + iy)| \leq ||F||_\infty e^{a \cdot z} \]
for all $x$ in $\hat{C}^0$ and $y$ in $\hat{G}$.

(ii) \[ \lim_{x \to 0} \int_D |F_0(x + iy) - F(y)|^2dy = 0 \]
where $x \to 0$ in $\hat{C}^0$ and $D$ is an arbitrary compact subset of the spine of $\Gamma$.

It is easy to see that the theorem follows, by translation, from the case $a = 0$. The details of this reduction are given in the following

**Lemma 3.1.** If the theorem is true for $a = 0$ it is true in general.

Suppose $F \in L_\infty(\hat{G})$ and that the additional hypotheses of the theorem are satisfied. Put $H(x) = e^{-i(a \cdot x)}F(x)$ and suppose $t \in C'$. Then $t - a \in (C - a)'$ and there exists $K$ in $L_1(\hat{G})$ such that $K * F \equiv 0$ and $k(t - a) \neq 0$ where $k(u) = (2\pi)^{-n/2}\int e^{i(u \cdot x)}K(x)dx$. Putting $L(x) = e^{-i(a \cdot x)}K(x)$ it follows that $1(t) = k(t - a) \neq 0$ and that $L * H(x) = e^{-i(a \cdot x)}K * F(x) = 0$. Assume the theorem is true for the case $a = 0$, and let $H_0$ be the extension of $H$. Define $F_0$ by

\[ F_0(x + iy) = e^{a \cdot (x + iy)}H_0(x + iy) \]
for $x \in \hat{C}^0$ and $y \in \hat{G}$. Then \[ |F_0(x + iy)| \leq e^{a \cdot z}||H||_\infty = e^{a \cdot z}||F||_\infty, \]
and if $D$ is a compact subset of the spine of $\Gamma$ we have,

\[ \int_D |F_0(x + iy) - F(y)|^2dy = \int_D |e^{a \cdot (x + iy)}H_0(x + iy) - e^{ia \cdot y}H(y)|^2dy \]
\[ \leq \int_D |e^{a \cdot z}H_0(x + iy) - H(iy)|^2dy. \]

Now $\int_D |e^{a \cdot z}H_0(x + iy) - H_0(x + iy)|^2dy = (e^{a \cdot z} - 1)^2\int_D |H_0(x + iy)|^2dy \to 0$ as $x \to 0$ in $\hat{C}^0$, in view of (ii), and the fact that $(e^{a \cdot z} - 1)^2 \to 0$. Combining these estimates we see that $\int_D |F_0(x + iy) - F(y)|^2dy \to 0$ as $x \to 0$ through values in $\hat{C}^0$.

**Proof of the theorem.** By the preceding lemma we can assume $a = 0$. Now let $T$ be the bounded operator on $L_2(\hat{G})$ given by the equation $T = U^{-1}M_FU$. It is apparent from (3.1) that it suffices to show that $T$ is causal with respect to the cone $C$. Since $T$ commutes with translations we need only show that $Tg$ vanishes outside $C$ whenever $g$ is in $L_2$ and vanishes outside $C$. Furthermore as $T$ is...
bounded it suffices, by continuity, to consider the case that \( g \) vanishes outside a compact subset \( N \) of \( C \). Now our assumptions clearly imply that the spectrum \( E \) of \( F \) is contained in \( C \), and, by Theorem 2.1, \( T \) is therefore dependent upon \( E \). Hence \( Tg \) vanishes outside \( E+N \). Finally since \( C \) is closed under addition, \( E+N \) is contained in \( C \) which concludes the proof.

References


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