ORDER AND COMMUTATIVITY IN BANACH ALGEBRAS

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S. Sherman has shown [4] that if the self adjoint elements of a C^* algebra form a lattice under their natural ordering the algebra is necessarily commutative. In this note we extend this result to real Banach algebras with an identity and arbitrary Banach * algebras with an identity. The central fact for a real Banach algebra A is that if the positive cone is defined to be the uniform closure of the set of finite sums of squares of elements of A, and if A is a lattice under the ordering induced by this cone, then extreme points of the unit sphere of the dual cone are multiplicative linear functionals. A similar situation holds for * algebras.

1. **Real Banach algebras.** Let X be a real Banach space, and let C be a closed cone in X.[1] For $x, y \in X$ we define $x \geqq y$ if $x - y \in C$. If in addition X is a lattice under the ordering $\geqq$, we say C lattice-orders X. Let C' be the dual cone and let $\sum = \{f \in C' : \|f\| \leqq 1\}$. The set of extreme points of $\sum$ will be denoted by S. For a real linear functional f let $I_f = \{x \in X : f(x) = 0\}$ and let $R = \bigcap_{f \in C'} I_f$. Lastly if X is a lattice we define $x_+ = x \vee 0$, $x_- = x \wedge 0$, and $|x| = x_+ - x_-$. We note $|x| \geqq 0$.

LEMMA 1. *If C is a closed cone in a real Banach space X, then*
(i) $R = C \cap -C$,
(ii) $R = \bigcap_{f \in S} I_f$,
(iii) *If C lattice-orders X,* $R = \{0\}$.

PROOF. Obviously $C \cap -C \subset R$. For the converse, by the Hahn-Banach theorem $x \in C$ iff $f(x) \geqq 0$ for each $f \in C'$. Therefore $R \subset C \cap -C$. For (ii) suppose $x \in \bigcap_{f \in S} I_f$, $x \notin R$, then there exists an $f \in C'$, $\|f\| = 1$, such that $|f(x)| = 2\epsilon \neq 0$. But by the Krein-Milman theorem there exist finitely many $f_i \in S$ and real numbers α_i such that $|f(x) - \sum \alpha_i f_i(x)| < \epsilon$. Hence for some i, $f_i(x) \neq 0$, which is a contradiction. Lastly let C lattice order X. Then $x \in C$ implies $x \geqq 0$ or $x_- = 0$, and $x \in -C$ implies $-x \geqq 0$ or $x_+ = 0$. Since $x = x_+ + x_-$, $x \in C \cap -C$ implies $x = 0$.

The central tool in both this investigation and that of Sherman is the following result of Krein and Krein [3]. It can be stated in a

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[1] We refer the reader to [2] for the appropriate definitions of cone, dual cone, etc.

slightly more general fashion, but the following is sufficient for our purposes.

THEOREM 1. *Let X be a real Banach space which is lattice ordered by a cone C. Suppose in addition that C contains an element e, $\|e\|=1$, such that $\{y: \|e-y\| \leqq 1\} \subset C$. Then $f \in S$ iff $|f(x)| = f(|x|)$ for each $x \in X$.*

Let us specialize to a real Banach algebra A with identity such that $\|1\|=1$, and let C be the closure of the set of finite sums of squares of elements of A. By the familiar binomial series argument (c.f. [2]), $\{y: \|1-y\| \leqq 1\} \subset C$. Also, for $f \in C'$ and $x, y \in A$ we have the Schwartz inequality, $[f(xy+yx)]^2 \leqq 4f(x^2)f(y^2)$, which may be verified by the classical argument. Also useful is the following property of functionals in C'.

LEMMA 2. *Let A be a real Banach algebra, and let $x \in C, f \in C'$. Then $f(x)=0$ implies $f(x^2)=0$.*

PROOF. First assume $x=y^2$, and let $f \in C'$. If $\|x\| \leqq 1$, the binomial series for $(1-x)^{1/2}$ converges absolutely. Therefore $1-x \in C$. Moreover since $x=y^2$, $y(1-x)^{1/2}=(1-x)^{1/2}y$ and $x(1-x)=[y(1-x)^{1/2}]^2 \in C$. Therefore $x-x^2 \geqq 0$. Hence $f(x)=0$ implies $f(x^2)=0$. We proceed now by induction. Let $x=\sum_{i=1}^{n+1} y_i^2$ and let $f(x)=0$. If $z=\sum_{i=1}^{n} y_i^2$, then $f(y_{n+1}^2)=f(z)=0$. We assume $f(z^2)=0$, and by the above argument $f(y_{n+1}^4)=0$. An application of the Schwartz inequality gives us

$$0 \leqq f(x^2) = f((z+y_{n+1}^2)^2) = f(zy_{n+1}^2 + y_{n+1}^2 z) \leqq 2[f(z^2)f(y_{n+1}^4)]^{1/2} = 0.$$

Therefore the result holds for all finite sums of squares, and by continuity it holds for all x in C.

THEOREM 2. *If C lattice-order A then each $f \in S$ is a homomorphism of A onto the real numbers, and A is commutative.*

PROOF. For $x, y \in A$ define the Jordan multiplication $x \circ y = (xy+yx)/2$. Thus A can be considered as a Jordan ring with an identity. We assert that for $f \in S$, I_f is a Jordan ideal. By Theorem 1 $x \in I_f$ iff $x_+, x_- \in I_f$. Therefore let $x \geqq 0$, $x \in I_f$. By the Schwartz inequality and Lemma 2, for any $y \in A$, $[f(xy+yx)]^2 \leqq 4f(x^2)f(y^2)=0$. Hence $xy+yx \in I_f$, and since I_f is obviously closed under addition, I_f is a Jordan ideal.

Now a linear functional of any algebra over a field which takes the identity of the algebra into the identity of the field is a homomorphism if its kernel is a two-sided ideal. Hence f is a Jordan homomorphism of A onto the reals. On the other hand Jacobson and Rickart [1, Theorem 2] have proved that a Jordan homomorphism of a ring into an integral domain is either a homomorphism or an anti-homomorphism. An application of this result proves that f is a homomorphism.

Finally for each $f \in S$ and $x, y \in A$, $xy - yx \in I_f$. Since by Lemma 1 $\bigcap_{f \in S} I_f = \{0\}$, A must be commutative.

2. **Banach * algebras.** Let A be a Banach * algebra with a continuous involution and an identity. Let C be the closure of the set of finite sums of elements xx^*. C is a closed cone in the real linear space H of self adjoint elements of A. The dual cone of C (in the conjugate space of H) can be identified with the set of those functionals f on A for which $f(xx^*) \geqq 0$ for each $x \in A$ (c.f. [2] for details). Let $\sum$, S be as before and for $f \in C'$ let $I_f = \{x \in A : f(x) = 0\}$ and $R = \bigcap_{f \in C'} I_f$. We also note that $\{h \in H : \|1 - h\| \leqq 1\} \subset C$ and for $f \in C'$ the familiar Schwartz inequality holds, i.e. $|f(xy^*)|^2 \leqq f(xx^*)f(yy^*)$, $x, y \in A$.

Lemma 3.

$$R = C \cap -C + i(C \cap -C),$$

$$R = \bigcap_{f \in S} I_f.$$

Proof. Let $T = (C \cap -C) + i(C \cap -C)$. Obviously $T \subset R$. If $x \in R$, let $h = (x + x^*)/2$, $k = (x - x^*)/2i$. Then h, k are self adjoint, $h, k \in I_f$ and $x = h + ik$. But a self adjoint element $y \in C$ iff $f(y) \geqq 0$ for each $f \in C'$. Therefore $h, k \in C \cap -C$ and $T = R$. For the second assertion if $x \in \bigcap_{f \in S} I_f$ and $x \notin R$, we may assume x is self adjoint and apply the argument of Lemma 1.

Lemma 4. *Let $h \in C$ and $f \in C'$. Then $f(h) = 0$ implies $f(h^2) = 0$.*

Proof. If $h = h^*$, and $\|h\| \leqq 1$, then by the familiar series argument $1 - h = k^2$, where $k = k^*$. Therefore $1 - h \in C$. Since $kh = hk$ and $khk \in C$, $khk = hk^2 = h - h^2 \in C$. Therefore $f(h) = 0$ implies $f(h^2) = 0$.

Theorem 3. *If C lattice-orders H, then each $f \in S$ is a homomorphism of A onto the complex numbers, and A is commutative.*

Proof. To prove that f is a homomorphism it suffices to show that for $f \in S$, I_f is a two-sided ideal. Let $x \in I_f$, $y \in A$. We assert $xy \in I_f$. First we may assume x is self adjoint and by Theorem 1 we may assume $x \geqq 0$. But then applying the Schwartz inequality

$$|f(xy)|^2 \leqq f(x^2)f(y^*y) = 0.$$

This proves $xy\in A$ and similarly $yx\in A$. Since I_f is obviously closed under addition, it is a two sided ideal. An application of Lemma 3 proves that A is commutative.

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A NOTE ON VALUED LINEAR SPACES

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Banaschewski [1] has given a simple and elegant proof of Hahn's embedding theorem for ordered abelian groups. His method can be used to prove the author's generalization of Hahn's theorem [2, p. 11]. In this note we make use of Banaschewski's method to prove a special case of the author's theorem (which is also a generalization of Hahn's theorem) that has been proven by Gravett [3].

Let (L, Δ, d) be a valued linear space [3]. That is, L is a vector space over a division ring K, Δ is a linearly ordered set with minimum element θ, and d is a mapping of L onto Δ such that for all $x, y\in L$, $d(x)=\theta$ if and only if $x=0$, $d(x)=d(kx)$ for all $0\neq k\in K$, and $d(x+y) \leqq \mathrm{Max}\,[d(x), d(y)]$. For each $\delta\in\Delta$, let $C^\delta=\{x\in L: d(x)\leqq\delta\}$ and let $C_\delta=\{x\in L: d(x)<\delta\}$. Let W be the vector space of all mappings f of Δ into the join of the spaces C^δ/C_δ for which $f(\delta)\in C^\delta/C_\delta$ and $R_f=\{\delta\in\Delta: f(\delta)\neq C_\delta\}$ is an inversely well ordered set. W is a subspace of the unrestricted direct sum V of the C^δ/C_δ. W is also a valued linear space (W, Δ, d'), with $d'(f)$ the largest $\delta\in R(f)$.

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