ON NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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1. The purpose of this paper is to consider the existence of large zeros for solutions of a class of nonlinear second order differential equations considered recently by C. T. Taam [**2**; **3**; **4**]. Equations of this type occur in astrophysics in considering the equilibrium of a gaseous configuration in stellar space, in atomic physics in the form of the Fermi-Thomas equation, and in mechanics in the study of free vibration of a hard spring not subject to damping.

A solution will be an absolutely continuous real-valued function with an absolutely continuous derivative and satisfying the differential equation almost everywhere in the sense of Carathéodory [**5**]. All solutions considered are different from the identically zero solution. All coefficients are assumed to be real-valued, bounded, Lebesgue-measurable functions of a real variables x for $x \geqq 0$. For the existence and uniqueness of solutions see Chapters I, II of [**5**].

2. THEOREM 1. *Let the following conditions be satisfied*:

(i) $f_n(x)$ *has a positive lower bound for* $x \geqq 0$,

(ii) $f_i(x) \geqq 0$, $f_i'(x) \in \mathcal{L}(0, \infty)$, $i = 1, 2, \cdots, n$, $x \geqq 0$, $f_i'(x) \leqq 0$,

(iii) n *a positive integer greater than one*,

(iv) $\int_0^\infty \sum_{i=1}^n f_i(x) x^{2i-1} dx < \infty$.

Then,

$$y'' + \sum_{i=1}^{n} f_i(x) y^{2i-1} \leqq 0 \tag{1}$$

has no solutions $y(x)$ *with arbitrarily large positive zeros.*

PROOF. According to Taam [**2**], for every solution $y(x)$ of (1), $y(x)$ and $y'(x)$ are bounded on $0 \leqq x < \infty$. Now let $R(x)$ be an amplitude variable defined for solutions $y(x)$ of (1) by

$$R(x) = \left(\frac{y'^2}{2} + \sum_{i=1}^{n} f_i y^{2i}\right). \tag{2}$$

$R(x)$ is positive for $x \geqq 0$ and

$$R'(x) = \sum_{i=1}^{n} f_i' y^{2i} \leqq 0. \tag{3}$$

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Equations (2), (3) imply that for any solution $y(x)$ of (1), $y'(x)$ remains bounded as $x \to \infty$. Let us assume that $y(x)$, a solution of (1), has arbitrarily positive zeros

$$x_1, x_2, \cdots, x_j, x_{j+1}, \cdots \tag{4}$$

and let x_j be a zero for which $y'(x_j) > 0$. Let $\hat{x}_j$ be the first zero of $y'(x)$ on (x_j, x_{j+1}). Upon integrating (1) over the interval $(x_j, \hat{x}_j)$ we have

$$y'(\hat{x}_j) - y'(x_j) + \int_{x_j}^{\hat{x}_j} \sum_{i=1}^{n} f_i y^{2i-1} dx = 0, \tag{5}$$

or,

$$y'(x_j) = \int_{x_j}^{\hat{x}_j} \sum_{i=1}^{n} f_i y^{2i-1} dx. \tag{6}$$

Now $y'(x) > 0$ and $y''(x) < 0$ so $y(x)$ is concave downward on $(x_j, \hat{x}_j)$, thus $y'(x)$ is decreasing on $(x_j, \hat{x}_j)$. So on $(x_j, \hat{x}_j)$ we have,

$$0 \leqq y(x) \leqq y'(x_j)(x - x_j); \quad 0 \leqq [y(x)]^{2i-1} \leqq [y'(x_j)(x - x_j)]^{2i-1} \tag{7}$$

for $i = 1, 2, \cdots, n$. Now from (6) and (7) we have

$$y'(x_j) \leqq \int_{x_j}^{\hat{x}_j} \sum_{i=1}^{n} f_i [y'(x_j)(x - x_j)]^{2i-1} dx. \tag{8}$$

Since $y'(x_j)$ is positive and bounded on $(x_j, \hat{x}_j)$ then for some index k, $1 \leqq k \leqq n$, we have,

$$1 \leqq \frac{1}{y'(x_j)} \int_{x_j}^{\hat{x}_j} \sum_{i=1}^{n} f_i [y'(x_j)(x - x_j)]^{2i-1}\, dx, \tag{9}$$

or,

$$1 \leqq [y'(x_j)]^{k-1} \int_{x_j}^{\hat{x}_j} \sum_{i=1}^{n} x^{2i-1} f_i dx. \tag{10}$$

As the zeros of $y(x)$, a solution of (1), become arbitrarily large, and since $y'(x_j)$ is bounded as $x_j \to \infty$, then the right-hand side of (10) tends to zero. Hence we reach a contradiction and the theorem holds.

3. THEOREM 2. *Let the following conditions be satisfied*:
(i) $f_n(x)$ *has a positive lower bound for* $x \geqq 0$,
(ii) $f_i(x) \geqq 0$, $f_i'(x) \leqq 0$, $f_i'(x) \in \mathfrak{L}(0, \infty)$, $i = 1, 2, \cdots, n$, $x \geqq 0$,
(iii) n *a positive integer greater than one*,
(iv) $y(x)$ *be a solution of* (1) *with arbitrarily large positive zeros.*

Then

$$\int_0^\infty x \sum_{i=1}^{n} f_i(x)dx = \infty. \tag{11}$$

Let us assume that (11) fails to hold and then show that there exists a solution $y(x)$ of (1) such that $y'(\infty)=0$, $y(\infty)=1$ which implies that $y(x)$ is not oscillatory for large positive values of x. This is equivalent to showing the existence of a solution of the following integral equation

$$y(x) = 1 - \int_x^\infty (t - x) \sum_{i=1}^{n} f_i y^{2i-1} dt. \tag{12}$$

Making use of the method of successive approximation as indicated by Atkinson [1], a solution to (12) may be shown to exist. Thus we reach a contradiction to (iv).

References

1. F. V. Atkinson, *On second order nonlinear oscillations*, Pacific J. Math. vol. 5 (1955) pp. 643–647.
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5. Coddington and Levinson, *Theory of ordinary differential equations*, Chapters I, II, New York, 1955.

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