THE CLASS OF RECURSIVE FUNCTIONS

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In this note, we find the position of the predicate "α is recursive" in the Kleene arithmetical hierarchy.[1] The proof involves a use of topology; the key step is the application of Baire's category theorem to a suitable function space.

We use the notation of [1]. In addition, we write (Ux) for the quantifier "there exist infinitely many x."

We designate the set of natural numbers by $\boldsymbol{N}$ and the class of mappings of $\boldsymbol{N}$ into $\boldsymbol{N}$ by $\boldsymbol{N}^{\boldsymbol{N}}$. We consider $\boldsymbol{N}$ as a topological space with the discrete topology, and $\boldsymbol{N}^{\boldsymbol{N}}$ as a topological space with the product topology. We review some known facts about $\boldsymbol{N}^{\boldsymbol{N}}$.

I. Provided with a suitable metric, $\boldsymbol{N}^{\boldsymbol{N}}$ is a complete metric space. (For it is the product of a countable number of discrete spaces.)

II. The nonrecursive functions are dense in $\boldsymbol{N}^{\boldsymbol{N}}$. (For a nonrecursive function remains nonrecursive if its values at a finite number of arguments are changed.)

III. If $R(\alpha)$ is a recursive predicate, then $\hat{\alpha}(R(\alpha))$ is open and closed. (Since the complement of $\hat{\alpha}(R(\alpha))$ is $\hat{\alpha}(\overline{R}(\alpha))$, it is sufficient to prove $\hat{\alpha}(R(\alpha))$ open. This follows from the fact that if $R(\alpha)$, then $R(\beta)$ for any function β agreeing with α on a certain finite set of arguments.)

LEMMA.[2] *For every predicate $R(\alpha, x, y, z)$ there is a predicate $S(\alpha, x, y)$ recursive in R such that*

$$(x)(Ey)(z)\,R(\alpha, x, y, z) \equiv (Ux)(y)S(\alpha, x, y)$$

for all α.

PROOF. Define

$$\begin{aligned} C(x, \alpha) &= \hat{y}(Ez)\overline{R}(\alpha, x, y, z), \\ D(x, \alpha) &= \hat{w}(y)_{y<w}(y \in C(x, \alpha)), \\ E(v, \alpha) &= \bigcup_{x<v} D(x, \alpha). \end{aligned}$$

Then

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[1] This problem was suggested to the author by John Addison.

[2] This lemma was suggested to the author by a proof due to Hartley Rogers.

$$E(v, \alpha) \text{ is finite} \equiv (x)_{x<v}D(x, \alpha) \text{ is finite}$$
$$\equiv (x)_{x<v}(Ey)(y \notin C(x, \alpha))$$
$$\equiv (x)_{x<v}(Ey)(z)R(\alpha, x, y, z).$$

Hence

$$(1) \qquad (x)(Ey)(z)R(\alpha, x, y, z) \equiv (Uv)(x)_{x<v}(Ey)(z)R(\alpha, x, y, z) \equiv (Uv)E(v, \alpha) \text{ is finite}.$$

Now $E(v, \alpha)$ is clearly recursively enumerable in R uniformly in v and α. Hence by the enumeration theorem, there is an e such that

$$z \in E(v, \alpha) \equiv (Ew)T_2^{R,\alpha}(e, v, z, w).$$

We recall that for each v, z there is at most one w such that $T_2^{R,\alpha}(e, v, z, w)$. It readily follows that

$$(Uv)E(v, \alpha) \text{ is finite} \equiv (Ux)(y)(T_2^{R,\alpha}(e, (x)_0, (x)_1, (x)_2) \;\&\; lh(x) \leqq 3 \;\&\; ((y)_0 > (x)_1 \rightarrow \overline{T}_2^{R,\alpha}(e, (x)_0, (y)_0, (y)_1))).$$

Combining with (1), we get the lemma.

THEOREM. *The predicate "α is recursive" can be written in the form $(Ex)(y)(Ez)R(\alpha, x, y, z)$ with R recursive but cannot be written in the form $(x)(Ey)(z)R(\alpha, x, y, z)$ with R recursive.*

PROOF. The first part is immediate, since

$$\alpha \text{ is recursive} \equiv (Ex)(y)(Ez)(T_1(x, y, z) \;\&\; U(z) = \alpha(y)).$$

Now suppose "α is recursive" could be written in the form $(x)(Ey)(z)R(\alpha, x, y, z)$ with R recursive. Then by the lemma and the enumeration theorem we would have

$$(2) \qquad \alpha \text{ is recursive} \equiv (Ux)(y)\overline{T}_1^1(\bar{\alpha}(y), e, x)$$

for a suitable e. We shall show that this leads to a contradiction.

Let $A(\alpha) = \hat{x}(y)\overline{T}_1^1(\bar{\alpha}(y), e, x)$. If α is nonrecursive, then $A(\alpha)$ is finite by (2). Since there are only countably many recursive functions and only countably many finite sets of natural numbers, there are only countably many different sets $A(\alpha)$. Hence the sets $\hat{\alpha}(A(\alpha) = B)$ with B a set of natural numbers form a countable covering of $\boldsymbol{N}^{\boldsymbol{N}}$. By I above and Baire's category theorem, it follows that there is a set B such that, setting $\mathfrak{C} = \hat{\alpha}(A(\alpha) = B)$, $\overline{\mathfrak{C}}$ has an interior point.

Since the nonrecursive functions are dense, there is a nonrecursive function β in $\overline{\mathfrak{C}}$. Now for any x,

$$\hat{\alpha}(x \in A(\alpha)) = \hat{\alpha}(y)\overline{T}_1^1(\bar{\alpha}(y), e, x) = \bigcap_y \hat{\alpha}\overline{T}_1^1(\bar{\alpha}(y), e, x)$$

is closed by III above. Thus

$$x \in B \to \mathcal{C} \subseteq \hat{\alpha}(x \in A(\alpha)) \to \bar{\mathcal{C}} \equiv \hat{\alpha}(x \in A(\alpha)) \to x \in A(\beta).$$

Thus $B \subseteq A(\beta)$. Since $A(\beta)$ is finite, B is finite.

Let $k = \max B$. Since $\bar{\mathcal{C}}$ has an interior point, there is a sequence number $s > 0$ such that

(3) $$(Ey)(\bar{\alpha}(y) = s) \to \alpha \in \bar{\mathcal{C}}.$$

Set

$$\gamma(0) = s,$$
$$\gamma(n+1) = \mu w(\mathrm{Ext}(w, \gamma(n)) \ \& \ w > \gamma(n) \ \& \ T_1^1(w, e, k+n+1))$$

(where $\mathrm{Ext}(w, z)$ means that w and z are sequence numbers and that $(w)_i = (z)_i$ for all $i < lh(z)$). We show that γ is well defined. Assume $\gamma(n)$ is defined. Then $\gamma(n)$ is a sequence number; so $\gamma(n) = \bar{\delta}(y)$ for some δ and y. By (3), $\delta \in \bar{\mathcal{C}}$. Hence there is a σ in $\mathcal{C}$ such that $\bar{\sigma}(y) = \bar{\delta}(y) = \gamma(n)$. Now $k+n+1 \notin B = A(\sigma)$. Hence for some u,

$$T_1^1(\bar{\sigma}(u), e, k+n+1);$$

and we may suppose $u > y$. If $w = \bar{\sigma}(u)$, then $\mathrm{Ext}(w, \gamma(n))$ & $w > \gamma(n)$ & $T_1^1(w, e, k+n+1)$. Hence $\gamma(n+1)$ is defined.

Clearly γ is recursive. Let $\alpha(n) = (\gamma(n))_n \dot{-} 1$; then α is recursive. Now for each n, $\gamma(n+1) = \bar{\alpha}(y)$ for some y; so $T_1^1(\bar{\alpha}(y), e, k+n+1)$ and hence $k+n+1 \notin A(\alpha)$. It follows that $A(\alpha)$ is finite. But this contradicts (2).

References

1. S. C. Kleene, *Hierarchies of number-theoretic predicates*, Bull. Amer. Math. Soc. vol. 61 (1955) pp. 193–213.

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