MULTIPLICATIONS ON THE LINE

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Mostert and Shields characterized ordinary multiplication on $[0, \infty)$ and briefly considered multiplication on $E_1=(-\infty, \infty)$ [2]. However, a characterization of ordinary multiplication on $E_1$ has apparently not been given. In this note, such a characterization is presented. In addition, all other topological semigroups on $E_1$ are determined in which 0 and 1 play their usual roles and whose multiplications agree on $[0, \infty)$ with ordinary multiplication. It is an interesting corollary of these results that Faucett's characterization of ordinary multiplication on the closed unit interval carries over to $E_1$ [1].

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In general, $(E_1, \circ)$ denotes a topological semigroup on $E_1$ in which $\circ$ is a continuous associative multiplication. The ordinary product of $x$ and $y$ is written $xy$, and $(E_1, \cdot)$ denotes the semigroup on $E_1$ under ordinary multiplication. We always assume that if $x, y \in [0, \infty)$, then $x \circ y = xy$. In addition, we suppose that 0 and 1 act as zero and identity respectively for $(E_1, \circ)$. Recall, from [2], that an isomorphism is a semigroup isomorphism which is also a homeomorphism.

We shall want to refer to the following classes of topological semigroups on $E_1$. Except for $(E_1, \cdot)$, these turn out to be the only possible topological semigroups on $E_1$ subject to the above restrictions.

**Example 1.** For fixed $\alpha > 0$, define multiplication $x \circ y$ in $E_1$ as follows:

(i) for $x \in E_1$, $y \in [0, \infty)$, $x \circ y = xy$,
(ii) for $x \in [0, \infty)$, $y \in (-\infty, 0)$, $x \circ y = x^{\alpha} y$,
(iii) for $x, y \in (-\infty, 0)$, $x \circ y = 0$.

**Example 2.** Define multiplication $x \tau y$ in $E_1$ as follows:

(i) for $x \in [0, \infty)$, $y \in E_1$, $x \tau y = y \tau x = xy$,
(ii) for $x, y \in (-\infty, 0)$, $x \tau y = -(xy)$.

Hereafter we shall abbreviate $(0, \infty)$ to $P$ and let $P^- = [0, \infty)$, $N' = (-\infty, 0)$ and $N^- = (-\infty, 0]$. It should be remarked that no two of the semigroups of Example 1 are isomorphic. For suppose $\alpha$ and $\beta$ are two positive numbers and that $\nu_1$ and $\nu_2$ are the corresponding multiplications. Suppose $T: (E_1, \nu_1) \to (E_1, \nu_2)$ is an isomorphism. Now the elements in $P^-$ can

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be recognized in any of the semigroups in Example 1 since these are the only elements with square roots. Hence $T$ is an isomorphism on $P^-$ to itself. Therefore there is a number $\gamma > 0$ such that $T(i) = t^\gamma$ if $t \geq 0$. Now for any such $t$, we have
\[
T(tv_1(-1)) = T((-1)v_1t^\alpha) = T(-1)v_2T(t^\alpha) = T(-1)v_2(t^\alpha)^\gamma = T(-1)(t^\gamma).
\]
The last equality is justified since $T(-1) \in (-\infty, 0)$. On the other hand,
\[
T(i)v_2T(-1) = (t^\gamma)v_2T(-1) = T(-1)(t^\gamma)^\beta.
\]
Therefore $t^\gamma = t^\gamma$ and hence $\alpha = \beta$.

We now proceed to derive the main theorems. We first prove several lemmas which hold without further restrictions on $(E_1, \circ)$.

**Lemma 1.** For any pair of distinct elements $a, b \in N$, there are numbers $t, t' \in P$, either both in $(0, 1)$ or both in $(1, \infty)$, such that $a = t \circ b = b \circ t'$.

**Proof.** For any $x \in N$, right and left multiplications by $x$ are continuous functions which map 0 into 0 and 1 into $x$. Therefore every point in the interval $(x, 0)$ is the image of a number in $(0, 1)$. Now either $a \in (b, 0)$ or $b \in (a, 0)$. In the first case, take $x = b$ and obtain $t, t' \in (0, 1)$ such that $a = t \circ b = b \circ t'$. In the second case, take $x = a$ and obtain $t, t' \in (0, 1)$ such that $a = (t^{-1}) \circ b = b \circ (t'^{-1})$.

**Lemma 2.** If $x \in N$ and $x^{-1}$ exists then $x^{-1} \in N$.

**Proof.** If $x^{-1} \in P$ then $x = (x^{-1})^{-1} \in P$.

**Lemma 3.** If $x \in N$ and $t \in P$ then $x \circ t$ and $t \circ x \in N$.

**Proof.** If $x \circ t \in P^-$ and $t \in P$ then $x = (x \circ t) \circ t^{-1} \in P^-$. A similar statement holds if $t \circ x \in P^-$.

**Lemma 4.** If $x \in N$ and $t, t'$ are distinct elements of $P$ then $t \circ x \neq t' \circ x$ and $x \circ t \neq x \circ t'$.

**Proof.** We prove only that $t \circ x \neq t' \circ x$, and for this it is sufficient to prove that $t \neq 1$ implies $t \circ x \neq x$. Suppose, on the contrary, that $t \circ x = x$. Then $t^{-1} \circ x = x$ also. Hence we may as well assume $t < 1$. Now for each positive integer $n$, $t^n \circ x = x$. But $t^n \circ x \to 0$, which is a contradiction since $x \in N$.

**Lemma 5.** If all members of $P$ commute with some element $v \in N$, then $(E_1, \circ)$ is a commutative semigroup.

**Proof.** Suppose $t \circ v = v \circ t$ for all $t \in P$. For any $x, y \in N$ there are numbers $t_1, t_2 \in P$ such that $x = t_1 \circ v$ and $y = t_2 \circ v$. If $t \in P$ then
Lemma 6. There is a positive number $\alpha$ such that $t \circ x = x \circ t^\alpha$ for all $x \in N^-$, $t \in P^-$. 

Proof. Fix $a \in N$ and set $f(t) = a \circ t$ and $g(t) = t \circ a$ for $t \in P^-$. $f$ and $g$ are continuous functions from $P^-$ to $N^-$. By Lemma 1, they map onto $N^-$ and by Lemma 4, they are one-to-one. Thus $f^{-1}g$ is a continuous function from $P^-$ to itself which is actually multiplicative. For let $t, t' \in P^-$ and let $s = f^{-1}g(t)$ and $s' = f^{-1}g(t')$; that is, $t \circ a = a \circ s$ and $t' \circ a = a \circ s'$. Now $f^{-1}g(tt') = f^{-1}((tt') \circ a) = f^{-1}(t \circ (t' \circ a)) = f^{-1}(t \circ (a \circ s')) = f^{-1}((a \circ s) \circ s') = f^{-1}(a \circ (ss')) = ss' = f^{-1}g(t)f^{-1}g(t')$. Therefore there is a number $\alpha$ such that for all $t \in P$, $f^{-1}g(t) = t^\alpha$. Since $f^{-1}g$ is continuous at zero, $\alpha$ is positive. By the definitions of $f$ and $g$ we have $t \circ a = a \circ t^\alpha$ for all $t \in P^-$. Now let $x \in N^-$ be arbitrary. There is $t_1 \in P^-$ such that $x = t_1 \circ a$. Therefore $x = a \circ t_1^\alpha$ and for $t \in P^-$, $t \circ x = t \circ (t_1 \circ a) = (t_1) \circ a = a \circ (t_1)^\alpha = a \circ (t \circ t_1) = (a \circ t_1) \circ t = x \circ t$. This completes the proof of the lemma.

Theorem 1. If there is at least one pair of elements $x_1, x_2 \in N$ such that $x_1 \circ x_2 \in P$, then $(E_1, \circ)$ is isomorphic to $E_1$ under ordinary multiplication.

Proof. Let $x, y$ be an arbitrary pair of elements in $N$. Write $x = t_1 \circ x_1$ and $y = x_2 \circ t_2$ with $t_1, t_2 \in P$. Then $x \circ y = (t_1 \circ x_1) \circ (x_2 \circ t_2) = t_1 \circ (x_1 \circ x_2) \circ t_2 \in P$. Thus, the product of any pair of elements in $N$ belongs to $P$. In particular, $x \circ x \in P$, so there is $t \in P$ such that $(x \circ x) \circ t = t \circ (x \circ x) = 1$. Therefore, $x \circ (x \circ t) = (t \circ x) \circ x = 1$, so every element of $N$ has an inverse which belongs to $N$, by Lemma 2. Now consider the function $f(x) = x \circ x$. $f$ is a continuous function from $N$ into $P$ having the property that if $t$ belongs to range $f$ then so does $t^{-1}$. Hence 1 belongs to range $f$; that is, there is at least one $u \in N$ such that $u \circ u = 1$. If $x$ is any other element of $N$ then there are elements $t, t' \in P$, either both in $(0, 1)$ or both in $(1, \infty)$, such that $x = t \circ u = u \circ t'$. Hence $x \circ x = (t \circ u) \circ (u \circ t') = t \circ (u \circ u) \circ t' = tt' \neq 1$. In other words, $u$ is the only square root of 1 which belongs to $N$. It follows that $u$ commutes with all elements of $P$. For if $t \in P$ then $t \circ u \circ t^{-1} \in N$, and $(t \circ u \circ t^{-1}) \circ (t \circ u \circ t^{-1}) = 1$. Hence $t \circ u \circ t^{-1} = u$, or $t \circ u = u \circ t$. From this we infer that $(E_1, \circ)$ is a commutative semigroup by Lemma 5.
Now set $I(x) = x$ if $x \in P^{-}$, and $I(x) = -(u \circ x)$ if $x \in N$.

It is straightforward to show that $I: (E_{1}, \circ) \to (E_{1}, \cdot)$ is an isomorphism and a homeomorphism, and the details are omitted.

Next we consider the possibilities in case the hypothesis of Theorem 1 is not satisfied—that is, in case $N^{-}$ is a sub-semigroup of $(E_{1}, \circ)$.

**Theorem 2.** If $u \circ v = 0$ for some $u, v \in N$ then $x \circ y = 0$ for all $x, y \in N$ and $(E_{1}, \circ)$ is isomorphic to one of the semigroups $(E_{1}, v)$ of Example 1.

**Proof.** Suppose $u \circ v = 0$ and write $v = u \circ t$ with $t \in P$. Then $0 = u \circ v = u \circ (u \circ t) = (u \circ u) \circ t$, so $u \circ u = 0$. Now for any $x, y \in N$, we have $x = s \circ u$ and $y = u \circ s'$ for some $s, s' \in P$. Thus $x \circ y = (s \circ u) \circ (u \circ s') = s \circ (u \circ u) \circ s' = 0$.

By Lemma 6, there is $\alpha > 0$ such that $t \circ x = x \circ t^{\alpha}$ for all $x \in N^{-}$, $t \in P^{-}$. Once again take advantage of the fact that if $a$ is a fixed element of $N$ then every element $x$ of $N$ can be written $x = t \circ a = a \circ t^{\alpha}$ for some (unique) $t \in P$. Now define $T: E_{1} \to E_{1}$ by

$$T(x) = - t^{\alpha} \text{ if } x \in N, \quad \text{and} \quad T(x) = x \text{ if } x \in P^{-}.$$

Then $T$ is an isomorphism from $(E_{1}, \circ)$ onto the semigroup $(E_{1}, v)$ of Example 1 which corresponds to the present value of $\alpha$. Certainly $T$ is a homeomorphism onto $E_{1}$. Let $x, y \in E_{1}$. If both $x, y$ belong to $N$ or if both belong to $P^{-}$, then obviously $T(x \circ y) = T(x) \cdot T(y)$. Suppose $x \in N$ and $y \in P^{-}$ with $x = t \circ a$. Then $T(y \circ x) = T(y \circ (t \circ a)) = T((yt) \circ a) = -(yt)^{\alpha}$, while $T(y) \cdot T(x) = y \cdot (-t^{\alpha}) = -(y^{\alpha}t^{\alpha})$. On the other hand, $T(x \circ y) = -(ty^{\beta})^{\alpha} = -t^{\alpha}y$. Finally, $T(x) \cdot T(y) = (-t^{\alpha})^{\beta}y = -t^{\alpha}y$ and the proof of the theorem is complete.

**Theorem 3.** If $x, y \in N$ imply $x \circ y \in N$, then $(E_{1}, \circ)$ is isomorphic to the semigroup of Example 2.

**Proof.** For any $a, b \in N$, we may write $b = (a \circ a) \circ t$ and $b = t' \circ (a \circ a)$ for some $t, t' \in P$. Hence $b = a \circ (a \circ t)$ and $b = (t' \circ a) \circ a$, and $a \circ t$ and $t' \circ a$ belong to $N$. Thus the equations $a \circ x = b$ and $y \circ a = b$ always have solutions in $N$. Therefore $(N, \circ)$ is a group and hence possesses exactly one idempotent $e$ which is an identity. Since $0$ is a zero for $N^{-}$, $(N^{-}, \circ)$ is a topological semigroup with zero and identity and no other idempotents. It follows from [1, Theorem A] that $(N^{-}, \circ)$ is isomorphic to $[0, \infty)$ under ordinary multiplication. In particular, multiplication is commutative on $N$. Now for any $t \in P$, $(t \circ e) \circ t^{-1} = (t \circ e \circ e) \circ t^{-1} = (t \circ e) \circ (e \circ t^{-1}) = (e \circ t^{-1}) \circ (t \circ e) = e$, so $t \circ e = e \circ t$. Hence $(E_{1}, \circ)$ is a commutative semigroup. Now define $T$ as follows: If $x \in N$ then $x = t \circ e$ for some $t \in P$. Set...
\[ T(x) = -t \text{ if } x \in N \quad \text{and} \quad T(x) = x \text{ if } x \in P^-. \]

It is easy to check that \( T \) is an isomorphism between \((E_1, \circ)\) and the semigroup \((E_1, \tau)\) of Example 2. The proof of the theorem is complete.

Finally, note that the semigroup of Example 2 has the idempotent \(-1\), while all of the semigroups of Example 1 contain nilpotent elements. Hence, we have the following extension to \( E_1 \) of Faucett's characterization of ordinary multiplication on the closed unit interval [1].

**Corollary.** If \( S \) is a topological semigroup on \( E_1 \) which possesses a zero and identity and no other idempotents, and if \( S \) contains no nonzero nilpotent elements, then \( S \) is isomorphic to \( E_1 \) under ordinary multiplication.

**References**