MULTIPLICATIONS ON THE LINE

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Mostert and Shields characterized ordinary multiplication on $[0, \infty)$ and briefly considered multiplication on $E_1 = (-\infty, \infty)$ [**2**]. However, a characterization of ordinary multiplication on E_1 has apparently not been given. In this note, such a characterization is presented. In addition, all other topological semigroups on E_1 are determined in which 0 and 1 play their usual roles and whose multiplications agree on $[0, \infty)$ with ordinary multiplication. It is an interesting corollary of these results that Faucett's characterization of ordinary multiplication on the closed unit interval carries over to E_1 [**1**].

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In general, $(E_1, \circ)$ denotes a topological semigroup on E_1 in which $\circ$ is a continuous associative multiplication. The ordinary product of x and y is written xy, and $(E_1, \cdot)$ denotes the semigroup on E_1 under ordinary multiplication. We always assume that if $x, y \in [0, \infty)$, then $x \circ y = xy$. In addition, we suppose that 0 and 1 act as zero and identity respectively for $(E_1, \circ)$. Recall, from [**2**], that an iseomorphism is a semigroup isomorphism which is also a homeomorphism.

We shall want to refer to the following classes of topological semigroups on E_1. Except for $(E_1, \cdot)$, these turn out to be the only possible topological semigroups on E_1 subject to the above restrictions.

EXAMPLE 1. For fixed $\alpha > 0$, define multiplication $x\nu y$ in E_1 as follows:

(i) for $x \in E_1$, $y \in [0, \infty)$, $x\nu y = xy$,

(ii) for $x \in [0, \infty)$, $y \in (-\infty, 0)$, $x\nu y = x^\alpha y$,

(iii) for $x, y \in (-\infty, 0)$, $x\nu y = 0$.

EXAMPLE 2. Define multiplication $x\tau y$ in E_1 as follows:

(i) for $x \in [0, \infty)$, $y \in E_1$, $x\tau y = y\tau x = xy$,

(ii) for $x, y \in (-\infty, 0)$, $x\tau y = -(xy)$.

Hereafter we shall abbreviate $(0, \infty)$ to P and let $P^- = [0, \infty)$, $N = (-\infty, 0)$ and $N^- = (-\infty, 0]$.

It should be remarked that no two of the semigroups of Example 1 are iseomorphic. For suppose α and β are two positive numbers and that ν_1 and ν_2 are the corresponding multiplications. Suppose $T: (E_1, \nu_1) \rightarrow (E_1, \nu_2)$ is an iseomorphism. Now the elements in P^- can

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be recognized in any of the semigroups in Example 1 since these are the only elements with square roots. Hence T is an iseomorphism on P^- to itself. Therefore there is a number $\gamma > 0$ such that $T(t) = t^\gamma$ if $t \geqq 0$. Now for any such t, we have

$$\begin{aligned} T(t\nu_1(-1)) &= T((-1)\nu_1 t^\alpha) = T(-1)\nu_2 T(t^\alpha) = T(-1)\nu_2(t^\alpha)^\gamma \\ &= T(-1)(t^{\alpha\gamma}). \end{aligned}$$

The last equality is justified since $T(-1) \in (-\infty, 0)$. On the other hand,

$$T(t)\nu_2 T(-1) = (t^\gamma)\nu_2 T(-1) = T(-1)(t^\gamma)^\beta.$$

Therefore $t^{\alpha\gamma} = t^{\gamma\beta}$ and hence $\alpha = \beta$.

We now proceed to derive the main theorems. We first prove several lemmas which hold without further restrictions on $(E_1, \circ)$.

LEMMA 1. *For any pair of distinct elements $a, b \in N$, there are numbers $t, t' \in P$, either both in $(0, 1)$ or both in $(1, \infty)$, such that $a = t \circ b = b \circ t'$.*

PROOF. For any $x \in N$, right and left multiplications by x are continuous functions which map 0 into 0 and 1 into x. Therefore every point in the interval $(x, 0)$ is the image of a number in $(0, 1)$. Now either $a \in (b, 0)$ or $b \in (a, 0)$. In the first case, take $x = b$ and obtain $t, t' \in (0, 1)$ such that $a = t \circ b = b \circ t'$. In the second case, take $x = a$ and obtain $t, t' \in (0, 1)$ such that $a = (t^{-1}) \circ b = b \circ (t'^{-1})$.

LEMMA 2. *If $x \in N$ and x^{-1} exists then $x^{-1} \in N$.*

PROOF. If $x^{-1} \in P$ then $x = (x^{-1})^{-1} \in P$.

LEMMA 3. *If $x \in N$ and $t \in P$ then $x \circ t$ and $t \circ x \in N$.*

PROOF. If $x \circ t \in P^-$ and $t \in P$ then $x = (x \circ t) \circ t^{-1} \in P^-$. A similar statement holds if $t \circ x \in P^-$.

LEMMA 4. *If $x \in N$ and t, t' are distinct elements of P then $t \circ x \neq t' \circ x$ and $x \circ t \neq x \circ t'$.*

PROOF. We prove only that $t \circ x \neq t' \circ x$, and for this it is sufficient to prove that $t \neq 1$ implies $t \circ x \neq x$. Suppose, on the contrary, that $t \circ x = x$. Then $t^{-1} \circ x = x$ also. Hence we may as well assume $t < 1$. Now for each positive integer n, $t^n \circ x = x$. But $t^n \circ x \to 0$, which is a contradiction since $x \in N$.

LEMMA 5. *If all members of P commute with some element $v \in N$, then $(E_1, \circ)$ is a commutative semigroup.*

PROOF. Suppose $t \circ v = v \circ t$ for all $t \in P$. For any $x, y \in N$ there are numbers $t_1, t_2 \in P$ such that $x = t_1 \circ v$ and $y = t_2 \circ v$. If $t \in P$ then

$t \circ x = t \circ (t_1 \circ v) = (tt_1) \circ v = v \circ (tt_1) = v \circ (t_1 t) = (v \circ t_1) \circ t = (t_1 \circ v) \circ t = x \circ t$, so members of P commute with all members of N. On the other hand: $x \circ y = (t_1 \circ v) \circ (t_2 \circ v) = (t_1 \circ v) \circ (v \circ t_2) = t_2 \circ (t_1 \circ v) \circ v = t_2 \circ (v \circ t_1) \circ v = (t_2 \circ v) \circ (t_1 \circ v) = y \circ x$, so members of N commute with each other.

LEMMA 6. *There is a positive number α such that $t \circ x = x \circ t^\alpha$ for all $x \in N^-$, $t \in P^-$.*

PROOF. Fix $a \in N$ and set $f(t) = a \circ t$ and $g(t) = t \circ a$ for $t \in P^-$. f and g are continuous functions from P^- to N^-. By Lemma 1, they map onto N^- and by Lemma 4, they are one-to-one. Thus $f^{-1}g$ is a continuous function from P^- to itself which is actually multiplicative. For let $t, t' \in P^-$ and let $s = f^{-1}g(t)$ and $s' = f^{-1}g(s')$; that is, $t \circ a = a \circ s$ and $t' \circ a = a \circ s'$. Now $f^{-1}g(tt') = f^{-1}((tt') \circ a) = f^{-1}(t \circ (t' \circ a)) = f^{-1}(t \circ (a \circ s')) = f^{-1}((t \circ a) \circ s') = f^{-1}((a \circ s) \circ s') = f^{-1}(a \circ (ss')) = ss' = f^{-1}g(t)f^{-1}g(t')$. Therefore there is a number α such that for all $t \in P$, $f^{-1}g(t) = t^\alpha$. Since $f^{-1}g$ is continuous at zero, α is positive. By the definitions of f and g we have $t \circ a = a \circ t^\alpha$ for all $t \in P^-$.

Now let $x \in N^-$ be arbitrary. There is $t_1 \in P^-$ such that $x = t_1 \circ a$. Therefore $x = a \circ t_1^\alpha$ and for $t \in P^-$, $t \circ x = t \circ (t_1 \circ a) = (tt_1) \circ a = a \circ (tt_1)^\alpha = a \circ (t^\alpha \circ t_1^\alpha) = (a \circ t_1^\alpha) \circ t^\alpha = x \circ t^\alpha$. This completes the proof of the lemma.

THEOREM 1. *If there is at least one pair of elements $x_1, x_2 \in N$ such that $x_1 \circ x_2 \in P$, then $(E_1, \circ)$ is iseomorphic to E_1 under ordinary multiplication.*

PROOF. Let x, y be an arbitrary pair of elements in N. Write $x = t_1 \circ x_1$ and $y = x_2 \circ t_2$ with $t_1, t_2 \in P$. Then $x \circ y = (t_1 \circ x_1) \circ (x_2 \circ t_2) = t_1 \circ (x_1 \circ x_2) \circ t_2 \in P$. Thus, the product of any pair of elements in N belongs to P. In particular, $x \circ x \in P$, so there is $t \in P$ such that $(x \circ x) \circ t = t \circ (x \circ x) = 1$. Therefore, $x \circ (x \circ t) = (t \circ x) \circ x = 1$, so every element of N has an inverse which belongs to N, by Lemma 2. Now consider the function $f(x) = x \circ x$. f is a continuous function from N into P having the property that if t belongs to range f then so does t^{-1}. Hence 1 belongs to range f; that is, there is at least one $u \in N$ such that $u \circ u = 1$. If x is any other element of N then there are elements $t, t' \in P$, either both in $(0, 1)$ or both in $(1, \infty)$, such that $x = t \circ u = u \circ t'$. Hence $x \circ x = (t \circ u) \circ (u \circ t') = t \circ (u \circ u) \circ t' = tt' \neq 1$. In other words, u is the only square root of 1 which belongs to N. It follows that u commutes with all elements of P. For if $t \in P$ then $t \circ u \circ t^{-1} \in N$, and $(t \circ u \circ t^{-1}) \circ (t \circ u \circ t^{-1}) = 1$. Hence $t \circ u \circ t^{-1} = u$, or $t \circ u = u \circ t$. From this we infer that $(E_1, \circ)$ is a commutative semigroup by Lemma 5.

Now set $I(x)=x$ if $x\in P^-$, and $I(x)=-(u\circ x)$ if $x\in N$.

It is straightforward to show that $I\colon (E_1, \circ)\to(E_1, \cdot)$ is an isomorphism and a homeomorphism, and the details are omitted.

Next we consider the possibilities in case the hypothesis of Theorem 1 is not satisfied—that is, in case N^- is a sub-semigroup of $(E_1, \circ)$.

THEOREM 2. *If $u\circ v=0$ for some u, $v\in N$ then $x\circ y=0$ for all x, $y\in N$ and $(E_1, \circ)$ is iseomorphic to one of the semigroups (E_1, ν) of Example* 1.

PROOF. Suppose $u\circ v=0$ and write $v=u\circ t$ with $t\in P$. Then $0=u\circ v=u\circ(u\circ t)=(u\circ u)\circ t$, so $u\circ u=0$. Now for any x, $y\in N$, we have $x=s\circ u$ and $y=u\circ s'$ for some s, $s'\in P$. Thus $x\circ y=(s\circ u)\circ(u\circ s')=s\circ(u\circ u)\circ s'=0$.

By Lemma 6, there is $\alpha>0$ such that $t\circ x=x\circ t^\alpha$ for all $x\in N^-$, $t\in P^-$. Once again take advantage of the fact that if a is a fixed element of N then every element x of N can be written $x=t\circ a=a\circ t^\alpha$ for some (unique) $t\in P$. Now define $T\colon E_1\to E_1$ by

$$T(x) = -t^\alpha \text{ if } x\in N, \quad\text{and}\quad T(x) = x \text{ if } x\in P^-.$$

Then T is an iseomorphism from $(E_1, \circ)$ onto the semigroup (E_1, ν) of Example 1 which corresponds to the present value of α. Certainly T is a homeomorphism onto E_1. Let x, $y\in E_1$. If both x, y belong to N or if both belong to P^-, then obviously $T(x\circ y)=T(x)\nu T(y)$. Suppose $x\in N$ and $y\in P^-$ with $x=t\circ a$. Then $T(y\circ x)=T(y\circ(t\circ a))=T((yt)\circ a)=-(yt)^\alpha$, while $T(y)\nu T(x)=y\nu(-t^\alpha)=-(y^\alpha t^\alpha)$. On the other hand, $T(x\circ y)=T((t\circ a)\circ y)=T(t\circ(a\circ y))=T(t\circ(y^\beta\circ a))$ where $\beta\alpha=1$. Thus $T(x\circ y)=-(ty^\beta)^\alpha=-t^\alpha y$. Finally, $T(x)\nu T(y)=(-t^\alpha)\nu y=-t^\alpha y$ and the proof of the theorem is complete.

THEOREM 3. *If x, $y\in N$ imply $x\circ y\in N$, then $(E_1, \circ)$ is iseomorphic to the semigroup of Example* 2.

PROOF. For any a, $b\in N$, we may write $b=(a\circ a)\circ t$ and $b=t'\circ(a\circ a)$ for some t, $t'\in P$. Hence $b=a\circ(a\circ t)$ and $b=(t'\circ a)\circ a$, and $a\circ t$ and $t'\circ a$ belong to N. Thus the equations $a\circ x=b$ and $y\circ a=b$ always have solutions in N. Therefore $(N, \circ)$ is a group and hence possesses exactly one idempotent e which is an identity. Since 0 is a zero for N^-, $(N^-, \circ)$ is a topological semigroup with zero and identity and no other idempotents. It follows from [1, Theorem A] that $(N^-, \circ)$ is iseomorphic to $[0, \infty)$ under ordinary multiplication. In particular, multiplication is commutative on N. Now for any $t\in P$, $(t\circ e)\circ t^{-1}=(t\circ e\circ e)\circ t^{-1}=(t\circ e)\circ(e\circ t^{-1})=(e\circ t^{-1})\circ(t\circ e)=e$, so $t\circ e=e\circ t$. Hence $(E_1, \circ)$ is a commutative semigroup. Now define T as follows: If $x\in N$ then $x=t\circ e$ for some $t\in P$. Set

$$T(x) = -t \text{ if } x \in N \quad \text{and} \quad T(x) = x \text{ if } x \in P^{-}.$$

It is easy to check that T is an iseomorphism between $(E_1, \circ)$ and the semigroup (E_1, τ) of Example 2. The proof of the theorem is complete.

Finally, note that the semigroup of Example 2 has the idempotent -1, while all of the semigroups of Example 1 contain nilpotent elements. Hence, we have the following extension to E_1 of Faucett's characterization of ordinary multiplication on the closed unit interval [1].

COROLLARY. *If S is a topological semigroup on E_1 which possesses a zero and identity and no other idempotents, and if S contains no nonzero nilpotent elements, then S is iseomorphic to E_1 under ordinary multiplication.*

REFERENCES

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