ON MEANS OF ENTIRE FUNCTIONS (II)[1]

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Let $f(z)$ be an entire function of order ρ and lower order λ. For δ $(0<\delta<\infty)$ and κ $(-1\leqq\kappa<\infty)$, let

$$M_\delta(r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta\right)^{1/\delta},$$

$$\mathfrak{M}_{\delta,\kappa}(r) = \frac{1}{r^{\kappa+1}}\int_0^r x^\kappa M_\delta(x)dx.$$

The following results can be easily verified [3, Lemmas 1 and 2].

If $M(r)$ denotes the maximum modulus of $f(z)$ for $|z| = r < R$, then

$$M_\delta(r) \leqq M(r) \leqq \left(\frac{R+r}{R-r}\right)^{1/\delta} M_\delta(R), \tag{1}$$

$$\limsup_{r\to\infty} \frac{\log\log M_\delta(r)}{\log r} = \limsup_{r\to\infty} \frac{\log\log \mathfrak{M}_{\delta,\kappa}(r)}{\log r} = \rho \quad (0\leqq\rho\leqq\infty),$$
(2)
$$\liminf_{r\to\infty} \frac{\log\log M_\delta(r)}{\log r} = \liminf_{r\to\infty} \frac{\log\log \mathfrak{M}_{\delta,\kappa}(r)}{\log r} = \lambda \quad (0\leqq\lambda\leqq\infty).$$

It is also known [1] that for a fixed δ, $\log M_\delta(r)$ is an increasing convex function of $\log r$. For a fixed r, $M_\delta(r)$ tends to $M(r)$ as δ tends to infinity.

Here we deduce the following:

THEOREM 1. *For an entire function $f(z)$ of finite order ρ*

$$\log M_\delta(r) \sim \log M(r) \qquad (0<\delta<\infty).$$

THEOREM 2. *If $0<\lambda<\rho<\infty$ is satisfied, then*

$$\log \mathfrak{M}_{\delta,\kappa}(r) \sim \log M_\delta(r) \qquad (0<\delta<\infty;\ -1\leqq\kappa<\infty).$$

PROOF OF THEOREM 1. Since $\log M_\delta(r)$ is a convex function of $\log r$, we have

$$\log M_\delta(r) = \log M_\delta(r_0) + \int_{r_0}^r \frac{m_\delta(x)}{x}\,dx, \tag{3}$$

where $m_\delta(x)$ is a nondecreasing function of x. $f(z)$ being of finite order ρ, we have for an arbitrary $\epsilon>0$

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$$\log M_\delta(r) < r^{\rho+\epsilon},$$

and therefore according to (3)

$$\int_r^{2r} \frac{m_\delta(x)}{x}\, dx < (2r)^{\rho+\epsilon}$$

and since $m_\delta(x)$ is nondecreasing

$$m_\delta(r) \log 2 < (2r)^{\rho+\epsilon}.$$

ϵ being arbitrary, we get $m_\delta(r) < r^{\rho+\epsilon}$. From (3) we have

$$\log M_\delta(R) = \log M_\delta(r) + \int_r^R \frac{m_\delta(x)}{x}\, dx$$

where

$$\begin{aligned}\int_r^R \frac{m_\delta(x)}{x}\, dx &< m_\delta(R) \log \frac{R}{r} \\ &= m_\delta(R) \log \left(1 + \frac{R-r}{r}\right) \\ &< R^{\rho+\epsilon} \cdot \frac{R-r}{r} \\ &= r^{\rho+\epsilon}\left(1 + \frac{R-r}{r}\right)^{\rho+\epsilon} \cdot \frac{R-r}{r}.\end{aligned}$$

Now choosing R subject to the condition

$$\frac{R-r}{r} = \frac{1}{r^{\rho+\epsilon}}$$

we get

$$\int_r^R \frac{m_\delta(x)}{x}\, dx < 2.$$

With this choice of R it follows from (1), that

$$\begin{aligned}\log M_\delta(r) &\leqq \frac{1}{\delta} \log \frac{R+r}{R-r} + \log M_\delta(R) \\ &< \frac{1}{\delta} \log (1 + 2r^{\rho+\epsilon}) + \log M_\delta(r) + 2 \\ &= (1 + o(1)) \log M_\delta(r).\end{aligned}$$

With $M_\delta(r) \leqq M(r)$, it gives

$$\log M(r) \sim \log M_\delta(r).$$

REMARK 1. From the method of proof it is clear that the theorem ($\log M(r) \sim \log M_\delta(r)$) is also true for the class of functions of infinite order for which for some finite κ

$$\frac{l_{\kappa+1}M(r)}{\log r} = \rho_\kappa$$

is finite but $\rho_{\kappa-1}$ is infinite, $l_\kappa x$ denotes the κth iterate of $\log x$, i.e. $l_1x = \log x$, $l_2x = \log \log x$, $l_3x = \log (l_2x)$ etc.

REMARK 2. For functions of finite order ρ and $1 \leqq \rho < \infty$ the theorem can be deduced more simply as follows.

If $f(z)$ is represented by the power series $\sum_{n=0}^{\infty} a_n z^n$, then

$$a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz$$

and we get

$$\mu(r) \leqq \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta = M_1(r) \tag{4}$$

where $\mu(r) = \max \{|a_0|, |a_1|r, |a_2|r^2, \cdots\}$. But $M_\delta(r)$ is a non-decreasing function of δ [2, pp. 14–18], therefore

$$\mu(r) \leqq M_1(r) \leqq M_\delta(r) \leqq M(r) \qquad (1 \leqq \delta < \infty).$$

For finite ρ, $\log \mu(r) \sim \log M(r)$ and we get the result.

PROOF OF THEOREM 2. Under the conditions imposed, the theorem follows immediately from the results

$$\limsup_{r\to\infty} \left\{\frac{M_\delta(r)}{\mathfrak{M}_{\delta,\kappa}(r)}\right\}^{1/\log r} = \rho, \qquad \liminf_{r\to\infty} \left\{\frac{M_\delta(r)}{\mathfrak{M}_{\delta,\kappa}(r)}\right\}^{1/\log r} = \lambda,$$

which follow on the same lines as the theorem in [3].

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