ON MEANS OF ENTIRE FUNCTIONS (II)¹

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Let \( f(z) \) be an entire function of order \( \rho \) and lower order \( \lambda \). For \( 0 < \delta < \infty \) and \( \kappa \) \((-1 \leq \kappa < \infty)\), let

\[
M_\delta(r) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta \right)^{1/\delta},
\]

\[
M_{\delta, \kappa}(r) = \frac{1}{r^{\kappa+1}} \int_0^r x^\kappa M_\delta(x) \, dx.
\]

The following results can be easily verified [3, Lemmas 1 and 2].

If \( M(r) \) denotes the maximum modulus of \( f(z) \) for \( |z| = r < R \), then

\[
M_\delta(r) \leq M(r) \leq \left( \frac{R+r}{R-r} \right)^{1/\delta} M_\delta(R),
\]

\[
\limsup_{r \to \infty} \frac{\log \log M_\delta(r)}{\log r} = \limsup_{r \to \infty} \frac{\log \log M_{\delta, \kappa}(r)}{\log r} = \rho \quad (0 \leq \rho \leq \infty),
\]

\[
\liminf_{r \to \infty} \frac{\log \log M_\delta(r)}{\log r} = \liminf_{r \to \infty} \frac{\log \log M_{\delta, \kappa}(r)}{\log r} = \lambda \quad (0 \leq \lambda \leq \infty).
\]

It is also known [1] that for a fixed \( \delta \), \( \log M_\delta(r) \) is an increasing convex function of \( \log r \). For a fixed \( r \), \( M_\delta(r) \) tends to \( M(r) \) as \( \delta \) tends to infinity.

Here we deduce the following:

**Theorem 1.** For an entire function \( f(z) \) of finite order \( \rho \)

\[
\log M_\delta(r) \sim \log M(r) \quad (0 < \delta < \infty).
\]

**Theorem 2.** If \( 0 < \lambda < \rho < \infty \) is satisfied, then

\[
\log M_{\delta, \kappa}(r) \sim \log M_\delta(r) \quad (0 < \delta < \infty; -1 \leq \kappa < \infty).
\]

**Proof of Theorem 1.** Since \( \log M_\delta(r) \) is a convex function of \( \log r \), we have

\[
\log M_\delta(r) = \log M_\delta(r_0) + \int_{r_0}^r \frac{m_\delta(x)}{x} \, dx,
\]

where \( m_\delta(x) \) is a nondecreasing function of \( x \). \( f(z) \) being of finite order \( \rho \), we have for an arbitrary \( \epsilon > 0 \)

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\[
\log M_\delta(r) < r^{\rho+e},
\]
and therefore according to (3)
\[
\int_r^{2r} \frac{m_\delta(x)}{x} \, dx < (2r)^{\rho+e}
\]
and since \(m_\delta(x)\) is nondecreasing
\[
m_\delta(r) \log 2 < (2r)^{\rho+e}.
\]
\(\epsilon\) being arbitrary, we get \(m_\delta(r) < r^{\rho+e}\). From (3) we have
\[
\log M_\delta(R) = \log M_\delta(r) + \int_r^R \frac{m_\delta(x)}{x} \, dx
\]
where
\[
\int_r^R \frac{m_\delta(x)}{x} \, dx < m_\delta(R) \log \frac{R}{r}
\]
\[
= m_\delta(R) \log \left(1 + \frac{R-r}{r}\right)
\]
\[
< R^{\rho+e} \cdot \frac{R-r}{r}
\]
\[
= r^{\rho+e} \left(1 + \frac{R-r}{r}\right)^{\rho+e} \cdot \frac{R-r}{r}.
\]
Now choosing \(R\) subject to the condition
\[
\frac{R-r}{r} = \frac{1}{r^{\rho+e}}
\]
we get
\[
\int_r^R \frac{m_\delta(x)}{x} \, dx < 2.
\]
With this choice of \(R\) it follows from (1), that
\[
\log M_\delta(r) \leq \frac{1}{\delta} \log \frac{R+r}{R-r} + \log M_\delta(R)
\]
\[
< \frac{1}{\delta} \log (1 + 2r^{\rho+e}) + \log M_\delta(r) + 2
\]
\[
= (1 + o(1)) \log M_\delta(r).
\]
With \( M_{\delta}(r) \leq M(r) \), it gives

\[
\log M(r) \sim \log M_{\delta}(r).
\]

**Remark 1.** From the method of proof it is clear that the theorem
\((\log M(r) \sim \log M_{\delta}(r))\) is also true for the class of functions of infinite
order for which for some finite \( \kappa \)

\[
\frac{l_{\kappa+1}M(r)}{\log r} = \rho_{\kappa}
\]
is finite but \( \rho_{\kappa-1} \) is infinite, \( l_{\kappa}x \) denotes the \( \kappa \)th iterate of \( \log x \), i.e.
\( l_1x = \log x \), \( l_2x = \log \log x \), \( l_3x = \log (l_2x) \) etc.

**Remark 2.** For functions of finite order \( \rho \) and \( 1 \leq \rho < \infty \) the theorem can be deduced more simply as follows.

If \( f(z) \) is represented by the power series \( \sum_{n=0}^{\infty} a_nz^n \), then

\[
a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} \, dz
\]
and we get

\[
(4) \quad \mu(r) \leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})| \, d\theta = M_1(r)
\]
where \( \mu(r) = \max \{|a_0|, |a_1| r, |a_2| r^2, \cdots \} \). But \( M_{\delta}(r) \) is a non-decreasing function of \( \delta \) [2, pp. 14–18], therefore

\[
\mu(r) \leq M_1(r) \leq M_{\delta}(r) \leq M(r) \quad (1 \leq \delta < \infty).
\]

For finite \( \rho \), \( \log \mu(r) \sim \log M(r) \) and we get the result.

**Proof of Theorem 2.** Under the conditions imposed, the theorem follows immediately from the results

\[
\limsup_{r \to \infty} \left\{ \frac{M_{\delta}(r)}{M_{\theta,\varepsilon}(r)} \right\}^{1/\log r} = \rho, \quad \liminf_{r \to \infty} \left\{ \frac{M_{\delta}(r)}{M_{\theta,\varepsilon}(r)} \right\}^{1/\log r} = \lambda,
\]
which follow on the same lines as the theorem in [3].

**References**


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