ON A CLASS OF UNIVALENT, STAR SHAPED MAPPINGS

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1. Introduction. Among all functions $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ regular and univalent in the unit circle, two classes of functions have been discussed extensively: The class of functions mapping the unit circle onto star shaped regions, characterized by $\operatorname{Re}\{zf'(z)/f(z)\} \geqq 0$ for $|z|<1$, and the class of functions mapping the unit circle onto convex regions, characterized by $\operatorname{Re}\{zf''(z)/f'(z)\}+1 \geqq 0$ for $|z|<1$.

This short paper will examine some of the geometric and analytic properties of a class of functions $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which map the unit circle onto a region whose geometric nature is somewhat intermediate between star shaped and convex. The functions under consideration are to satisfy $\operatorname{Re}\{zf'(z)/f(z)\} \geqq 1/2$ for all $|z|<1$.

Interest in functions of this type can be traced back to two papers by A. Marx [4] and E. Strohäcker [8] who have shown that for any function $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, which maps the unit circle onto a convex region, we have $\operatorname{Re}\{zf'(z)/f(z)\} \geqq 1/2$, and as the function $f(z)=z/(1+z)$ shows, the constant 1/2 cannot be improved. It is also clear that the converse is not true, i.e. functions for which $\operatorname{Re}\{zf'(z)/f(z)\} \geqq 1/2$ need not map the unit circle onto a convex region. An example of this type is given by the function $w=f(z)=z-1/3\, z^2$ for which $\operatorname{Re}\{zf'(z)/f(z)\} \geqq 1/2$, $|z| \leqq 1$, yet the image region is not convex.

Recently, interest in this class of functions was roused again in a paper by R. F. Gabriel [2]. There it is shown that if $p(z)$ is analytic and single valued for $|z|<1$, and if we denote by $w_1(z)=1+\sum_{n=2}^{\infty} a_n z^n$ and by $w_2(z)=z+\sum_{n=2}^{\infty} b_n z^n$ two linearly independent solutions of $w''+p(z)w=0$, then $f(z)=w_1(z)/w_2(z)=1/z+\cdots$ will map $|z| \leqq 1$ onto the exterior of a convex region if and only if $\operatorname{Re}\{zf'(z)/f(z)\} \geqq 1/2$ for $|z|<1$.

M. S. Robertson [6] considered also the class of functions $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ for which $\operatorname{Re}\{zf'(z)/f(z)\} \geqq \alpha>0$, $|z|<1$.

Notation. On the following pages we shall denote the class of functions $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ regular and univalent in the unit circle by S.

The subclass of S which map $|z|<1$ onto a star shaped region, i.e. for which $\operatorname{Re}\{zf'(z)/f(z)\} \geqq 0$ by St.

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The subclass of St which map $|z|<1$ onto a "special" star shaped region, i.e. for which $\operatorname{Re}\{zf'(z)/f(z)\} \geqq 1/2$ by St*.

The subclass of S which map $|z|<1$ onto a convex region, i.e. for which $\operatorname{Re}\{zf''(z)/f'(z)\}+1 \geqq 0$ by C.

By [4] and [8] we obviously have:

1.1 $$S \supset \mathrm{St} \supset \mathrm{St}^* \supset C.$$

2. Methods for constructing functions of class St*.

The intermediacy of St* between St and C suggests, that one might obtain functions of class St* by "relaxing" the conditions on functions of class C or by "strengthening" conditions on functions of class St. Both methods yield results.

THEOREM 2.1. *If $w=f(z)\in C$, then $g(z)=z(f'(z))^{\alpha}\in \mathrm{St}^*$ for $0<\alpha \leqq 1/2$. The converse is true for $\alpha=1/2$.*

PROOF. $\operatorname{Re}\{zg'(z)/g(z)\}=\operatorname{Re}\{1+\alpha\, zf''(z)/f'(z)\} \geqq 1-\alpha \geqq 1/2$. For the converse we have: $\operatorname{Re}\{zf''(z)/f'(z)\}+1=\operatorname{Re}\{2zg'(z)/g(z)\}-1 \geqq 0$. This should be compared with the well known result that if $f(z)\in C$ then $g(z)=zf'(z)\in \mathrm{St}$ and conversely.

COROLLARY 2.2. *Let $|b_i|=1$ and $\sum_{i=1}^{n}\mu_i \leqq 1$, $\mu_i \geqq 0$, then $w=g(z) =z\prod_{i=1}^{n}(1-b_iz)^{-\mu_i}\in \mathrm{St}^*$.*

PROOF. $f(z)=\int_0^z\prod_{i=1}^{n}(1-b_it)^{-v_i}dt\in C$ if $v_i\geqq 0$ and $\sum_{i=1}^{n} v_i \leqq 2$, $|b_i|=1$. The corollary follows if we let $\alpha=1/2$ in Theorem 2.1.

THEOREM 2.3. *If $w=f(z)\in \mathrm{St}$, then $g(z)=z\{f(z)/z\}^{\alpha}\in \mathrm{St}^*$ for $0<\alpha \leqq 1/2$. The converse is true for $\alpha=1/2$.*

PROOF. $\operatorname{Re}\{zg'(z)/g(z)\}=\operatorname{Re}\{\alpha zf'(z)/f(z)+1-\alpha\} \geqq 1-\alpha \geqq 1/2$.

For the converse we have: $\operatorname{Re}\{zf'(z)/f(z)\}=\operatorname{Re}\{2zg'(z)/g(z)-1\} \geqq 0$. (Marx [4] has the above theorem for $\alpha=1/2$.)

The method of Theorem 2.3 for constructing functions of class St* gives some insight into the geometrical nature of the region onto which the unit circle is mapped by functions of class St*. If $z=re^{i\theta}$ is mapped by $w=f(z)\in \mathrm{St}$ into $Re^{i\phi}$, then a comparison of r and R and θ and ϕ tells us about the amount of distortion. We notice that $g(z)=\{zf(z)\}^{1/2}\in \mathrm{St}^*$ reduces the amount of distortion effected by $f(z)\in \mathrm{St}$ by an "averaging" process, i.e. $z=re^{i\theta}$ will be mapped by $g(z)$ into $\rho e^{i\psi}$ where $\rho=(rR)^{1/2}$ and $\psi=(\theta+\phi)/2$.

3. Distortion theorems for functions of class St*.

THEOREM 3.1. *For all $g(z)\in \mathrm{St}^*$ we have $|z|/(1+|z|) \leqq |g(z)| \leqq |z|/(1-|z|)$.*

PROOF. By Theorem 2.3 ($\alpha=1/2$) and the "Verzerrungs Satz" we have: $|z|/(1+|z|)^2 \leqq |g^2(z)/z| \leqq |z|/(1-|z|)^2$ and hence the Theorem follows. These inequalities are sharp for $g(z)=z/(1+z)\in\text{St}^*$, $z=\pm r$.

THEOREM 3.2. *For all* $g(z)\in\text{St}^*$ *the domain of values of* $zg'(z)/g(z)$ *is the circle with center at* $1/(1-|z|^2)$ *and radius* $|z|/(1-|z|^2)$.

PROOF. Let

$$G(z) = zg'(z)/g(z) - 1/2$$

and

$$H(z) = (2G(z) - 1)/(2G(z) + 1) = \{zg'(z)/g(z) - 1\}/zg'(z)/g(z).$$

Then $H(z)$ is regular for $|z|<1$, $H(0)=0$ and $|H(z)|<1$ for $|z|<1$. Hence the Lemma of Schwarz can be applied and we have for $|z|<1$

$$\left|\left(\frac{zg'(z)}{g(z)} - 1\right)\Big/\frac{zg'(z)}{g(z)}\right| < |z| \quad \text{or:} \quad \left|\frac{zg'(z)}{g(z)} - 1\right| < |z|\left|\frac{zg'(z)}{g(z)}\right|.$$

But the domain defined by this inequality is the interior of the Circle of Apollonius with the line segment form $1/(1+|z|)$ to $1/(1-|z|)$ as a diameter, i.e. the interior of the circle with radius $|z|/(1-|z|^2)$ and center at $1/(1-|z|^2)$. The function $f(z)=z/(1+z)\in\text{St}^*$ shows that the theorem cannot be improved.

4. **Some coefficient relations.** It is well known that if $f(z)\in\text{St}$ then $|a_n|\leqq n$. For functions of class St* we have:

THEOREM 4.1. *If* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n\in\text{St}^*$, *then* $|a_n|\leqq 1$.

PROOF. Let $p(z)=2zf'(z)/f(z)-1=1+c_1z+c_2z^2+\cdots$. Since $p(z)$ is regular and $\text{Re}\{p(z)\}>0$ for $|z|<1$, therefore, by a well known lemma we have $|c_n|\leqq 2$ for $n=1, 2, 3, \cdots$.

Comparing coefficients we obtain $2(n-1)a_n=c_{n-1}+a_2c_{n-2}+\cdots+a_{n-1}c_1$, and hence: $|a_n|\leqq 1/(n-1)\{1+|a_2|+\cdots+|a_{n-1}|\}$. It follows now by induction that $|a_n|\leqq 1$ for $n=1, 2, 3, \cdots$.

The function $f(z)=z/(1-z)=\sum_{n=1}^{\infty} z^n\in\text{St}^*$ shows that these inequalities are sharp.

For some kind of a converse we have:

THEOREM 4.2. *If* $g(z)=z+\sum_{n=2}^{\infty} a_n z^n$ *and if* $\sum_{n=2}^{\infty}(2n-1)|a_n|\leqq 1$ *then* $g(z)\in\text{St}^*$.

PROOF. The proof is based on a method used by A. Goodman [**3**]. We have:

$$\frac{zg'(z)}{g(z)} - \frac{1}{2} = \frac{1 + 3a_2z + 5a_3z^2 + \cdots + (2n-1)a_nz^{n-1} + \cdots}{2(1 + a_2z + a_3z^2 + \cdots + a_nz^{n-1} + \cdots} \tag{4.3}$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty} b_nz^n$$

where

$$\begin{aligned} b_1 &= a_2. \\ b_2 &= 2a_3 - a_2b_1, \\ b_3 &= 3a_4 - a_2b_2 - a_3b_1, \\ &\vdots \\ b_{n-1} &= (n-1)a_n - a_2b_{n-2} - a_3b_{n-3} - \cdots - a_{n-1}b_1, \\ &\vdots \end{aligned} \tag{4.4}$$

and therefore for $n \geqq 2$

$$\sum_{k=1}^{n-1} b_k = \sum_{k=2}^{n} (k-1)a_k - a_2\sum_{k=1}^{n-2} b_k - a_3\sum_{k=1}^{n-3} b_k - \cdots - a_{n-1}b_1. \tag{4.5}$$

The inequality of the theorem implies that $|b_1| = |a_2| \leqq 1/3$. It will now be shown by mathematical induction that for all n we have $|\sum_{k=1}^{n} b_k| \leqq 1/2$. Assume that $|\sum_{k=1}^{m} b_k| \leqq 1/2$ for $m = 1, 2, 3, \cdots, n-2$. Then (4.5) yields

$$\left|\sum_{k=1}^{n-1} b_k\right| \leqq \sum_{k=2}^{n} (k-1)|a_k| + \frac{1}{2}\sum_{k=2}^{n-1} |a_k| \leqq \sum_{k=2}^{n} (k-1/2)|a_k| \tag{4.6}$$
$$= \frac{1}{2}\sum_{k=2}^{n} (2k-1)|a_k| \leqq 1/2$$

by the inequality of the theorem, and hence $|\sum_{k=1}^{n} b_k| \leqq 1/2$ for all n. From (4.3) it follows now that

$$|zg'(z)/g(z) - 1| \leqq 1/2 \text{ for } z = r. \tag{4.7}$$

But the inequality of Theorem 4.2 and the special starshapedness of the image domain are invariant under rotations of the z and w planes. Hence any point in the unit circle may be placed in the interval $[0, 1]$, and thus (4.7) is valid throughout the unit circle i.e.

$$\text{Re}\,\{zg'(z)/g(z)\} \geqq 1/2 \text{ for } |z| < 1.$$

COROLLARY 4.8. *Let* $g(z) = z - \sum_{n=2}^{\infty} a_nz^n$, *where all* $a_i \geqq 0$, *then* $g(z) \in \text{St}^*$, *if and only if* $\sum_{n=2}^{\infty} (2n-1)a_n \leqq 1$.

PROOF. The sufficiency of the condition follows from the Theorem. For the necessity we have:

$$\text{(4.9)} \qquad \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}-\frac{1}{2}\right\}=\operatorname{Re}\left\{\frac{1-\sum_{n=2}^{\infty}(2n-1)a_nz^{n-1}}{2\left(1-\sum_{n=2}^{\infty}a_nz^{n-1}\right)}\right\}.$$

If $\sum_{n=2}^{\infty}(2n-1)a_n>1$, then we could find a positive value of z, r_0, for which the numerator of (4.9) would be negative and the denominator positive and hence $\operatorname{Re}\{zg'(z)/g(z)\}<1/2$ for that value of z and all positive values of $z>r_0$. (See also [7].)

5. **The radius of special star shapedness.** We define the radius of special star shapedness, r^*, as the upper bound of the radii, r, of circles $|z|\leqq r$, which are mapped by any function $f(z)\in S$ onto a region of special star shapedness, i.e. that for all functions $f(z)\in S$ we have $\operatorname{Re}\{zf'(z)/f(z)\}\geqq 1/2$ for all $|z|\leqq r$. From the introduction it follows that r^* is larger than the bound of convexity (Rundungsschranke) and less than the bound of starlikeness. Therefore

$$\text{(5.1)} \qquad 2-3^{1/2}=.268\cdots\leqq r^*\leqq .65\cdots=\tanh \pi/4.$$

We have:

THEOREM 5.2. *$r^*=1/3$ for all $f(z)\in\mathrm{St}$, and hence $r^*\leqq 1/3$.*

PROOF. By [5] we have for any $f(z)\in\mathrm{St}$

$$\frac{1-|z|}{1+|z|}\leqq \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\leqq\frac{1+|z|}{1-|z|}.$$

Therefore $\operatorname{Re}\{zf'(z)/f(z)\}\geqq(1-|z|)/(1+|z|)\geqq 1/2$ for all $|z|\leqq 1/3$. This result is sharp for $f(z)=z(1+z)^{-2}\in\mathrm{St}$ and $z=1/3$.

THEOREM 5.3. *A lower bound for r^* is: $r^*>.301\cdots$.*

PROOF. If we let

$$\text{(5.4)} \qquad \begin{aligned} g(z)&=f^{-1/2}(z^{-2})\\ &=z+b_1/z+b_3/z^3+\cdots+b_{2n-1}/z^{2n-1}+\cdots \end{aligned}$$

then by Bieberbach's Flächensatz [1] we have

$$\text{(5.5)} \qquad \sum_{n=1}^{\infty}(2n-1)|b_{2n-1}|^2\leqq 1.$$

The condition $\operatorname{Re}\{zf'(z)/f(z)\}\geqq 1/2$ is equivalent to

$$\left|\frac{zf'(z)}{f(z)}\right| \geqq \left|\frac{zf'(z)}{f(z)} - 1\right| \tag{5.6}$$

i.e. by (5.4)

$$\left|\frac{z^{-1/2}g'(z^{-1/2})}{g(z^{-1/2})}\right| \geqq \left|\frac{z^{-1/2}g'(z^{-1/2})}{g(z^{-1/2})} - 1\right|$$

or:

$$\begin{aligned} &\left|\frac{1 - b_1 z - 3b_3 z^2 - \cdots - (2n-1)b_{2n-1}z^n - \cdots}{1 + b_1 z + b_3 z^2 + \cdots + b_{2n-1}z^n + \cdots}\right| \\ &\qquad \geqq \left|\frac{1 - b_1 z - 3b_3 z^2 - \cdots - (2n-1)b_{2n-1}z^n + \cdots}{1 + b_1 z + b_3 z^2 + \cdots + b_{2n-1}z^n + \cdots} - 1\right| \end{aligned} \tag{5.7}$$

or:

$$\begin{aligned} &| 1 - b_1 z - 3b_3 z^2 - \cdots - (2n-1)b_{2n-1}z^n | \\ &\qquad \geqq 2| b_1 z + 2b_3 z^2 + 3b_5 z^3 + \cdots + nb_{2n-1}z^n + \cdots |. \end{aligned}$$

A sufficient condition for this inequality to be satisfied is that

$$\begin{aligned} 1 - |b_1| r - 3|b_3| r^2 - \cdots - (2n-1)|b_{2n-1}| r^n + \cdots \\ \geqq 2|b_1| r + 4|b_3| r^2 + \cdots \end{aligned}$$

or:

$$\begin{aligned} 3|b_1| r + 7|b_3| r^2 + 11|b_5| r^3 + \cdots \\ + (4n-1)|b_{2n-1}| r^n + \cdots \leqq 1 \end{aligned}$$

or

$$\begin{aligned} &2\{ |b_1| r + 3|b_3| r^2 + \cdots + (2n-1)|b_{2n-1}| r^n + \cdots \} \\ &\qquad + \{ |b_1| r + |b_3| r^2 + \cdots + |b_{2n-1}| r^n + \cdots \} \leqq 1. \end{aligned} \tag{5.8}$$

If in the first parenthesis of inequality (5.8) we let $c_n = (2n-1)^{1/2}|b_{2n-1}|$ and $d_n = (2n-1)^{1/2} \cdot r^n$ then the Inequality of Schwarz

$$\sum c_n \cdot d_n \leqq \left(\sum c_n^2\right)^{1/2} \cdot \left(\sum d_n^2\right)^{1/2}$$

gives

$$\begin{aligned} &|b_1| r + 3|b_3| r^2 + \cdots + (2n-1)|b_{2n-1}| r^n + \cdots \\ &\qquad \leqq (|b_1|^2 + 3|b_3|^2 + \cdots + (2n-1)|b_{2n-1}|^2 + \cdots)^{1/2} \\ &\qquad \cdot (r^2 + \cdots + (2n-1)r^{2n} + \cdots)^{1/2} \end{aligned} \tag{5.9}$$

and therefore by (5.5)

$$(5.10)\quad \begin{aligned} &|b_1|r + 3|b_3|r^2 + \cdots + (2n-1)|b_{2n-1}|r^n + \cdots \\ &\leqq (r^2 + 3r^4 + \cdots + (2n-1)r^{2n} + \cdots)^{1/2} = \frac{r(1+r^2)^{1/2}}{1-r^2}. \end{aligned}$$

Similarly, if in the second parenthesis of inequality (5.8) we let $c_n=(2n-1)^{1/2}|b_{2n-1}|$, $d_n=r^n(2n-1)^{-1/2}$ and apply the Inequality of Schwarz again, we get:

$$(5.11)\quad \begin{aligned} &|b_1|r + |b_3|r^2 + \cdots + |b_{2n-1}|r^n + \cdots \\ &\leqq (|b_1|^2 + 3|b_3|^2 + \cdots + (2n-1)|b_{2n-1}| + \cdots)^{1/2} \\ &\cdot\left(r^2 + \frac{r^4}{3} + \cdots + \frac{r^{2n}}{2n-1} + \cdots\right)^{1/2} \end{aligned}$$

and again by (5.5)

$$(5.12)\quad \begin{aligned} &|b_1|r + |b_3|r^2 + \cdots + |b_{2n-1}|r^n + \cdots \\ &\leqq \left(r^2 + \frac{r^4}{3} + \cdots + \frac{r^{2n}}{2n-1} + \cdots\right)^{1/2} = \left(\frac{r}{2}\ln\frac{1+r}{1-r}\right)^{1/2}. \end{aligned}$$

Substituting (5.10) and (5.12) into (5.8) we have:

$$(5.13)\quad \frac{2r(1+r^2)^{1/2}}{1-r^2} + \left(\frac{r}{2}\log\frac{1+r}{1-r}\right)^{1/2} < 1.$$

This will be satisfied for $|z| = r < .301 \cdots$, i.e. $r^* > .301 \cdots$.

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