ASYMPTOTIC PROPERTIES OF SUBHARMONIC AND ANALYTIC FUNCTIONS

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1. **Introduction.** Concerning the regularity which occurs in the setting of Phragmén-Lindelöf principle, the theorem of Ahlfors and Heins [2] (see also [6]) states;

THEOREM 1. *If* $u(z)$ *is subharmonic in* $\Re z>0$ *and satisfies the Phragmén-Lindelöf boundary condition, and if* $\alpha\equiv\sup u/x<\infty$ $(x=\Re z>0)$, *then* $\lim_{r\to\infty} u(re^{i\theta})/r \cos\theta=\alpha$ *for* $|\theta|<\pi/2$ *with the exception at most of outer logarithmic capacity zero; uniformly in any closed interior angle if* r *is excluded from a set of finite logarithmic length; and without exception in any interior angle in which* $u(z)$ *is harmonic.*

Recently R. P. Boas, Jr. has studied how much the boundary condition can be weakened without destroying the conclusion for the functions of exponential type [3]. In this paper, we shall treat the generalizations and precisions of some results in [3].

THEOREM 2. *If* $u(z)$ *is subharmonic for* $\Re z\geqq 0$ *and satisfies*

(a) $$\liminf_{r\to\infty}\frac{1}{r}\int_{-\pi/2}^{+\pi/2}u^+(re^{i\theta})\cos\theta d\theta<\infty$$

and

(b) $$\int_{r_1}^{r_2}t^{-2}u(\pm it)dt<\epsilon \text{ for all } r_2>r_1>N$$

if N *is selected sufficiently large for an arbitrary positive* ϵ, *then* $\lim_{r\to\infty} u(re^{i\theta})/r\cos\theta$ *exists (a finite limit) for* $|\theta|<\pi/2$ *under the same exception as those of Theorem* 1.

Each of the conditions $\int_{-\infty}^{+\infty}(t^2+1)^{-1}u^+(it)dt<\infty$ and the existence of

$$\int_1^{+\infty}t^{-2}u(\pm it)dt$$

implies the condition (b). Accordingly, as two special cases of Theorem 2, we can obtain results of Boas [3, Theorem 2 and 3].

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THEOREM 3. *If* $u(z)$ *is subharmonic for* $\Re z \geqq 0$ *and satisfies the conditions* (a),

(c) $\int_{r_1}^{r_2} t^{-2}\{u(it) + u(-it)\}dt < \epsilon$ *for all* $r_2 > r_1 > N$ *if* N *is selected sufficiently large for an arbitrary positive* ϵ,

$$\int_{r}^{\lambda r} t^{-3}u(\epsilon it)dt = o(r^{-1}) \tag{d}$$

(λ; *an arbitrary real number such that* $2 \geqq \lambda > 1$)

and

$$\int_{\delta}^{r} t^{-3}u(\epsilon it)dt = o(r^{-1}) \tag{e}$$

(δ; *an arbitrary real number such that* $r \geqq \delta > 0$),

ϵ *being* $+1$ *or* -1, *then the conclusion of Theorem* 1 *holds.*

Let $f(z)$ be regular for $\Re z > 0$, and the zero points of $f(z)$ in a domain $D[|\arg z| < \pi/2,\ |z| > 1]$ denote by $\zeta_1, \zeta_2, \cdots$. We denote the circular disks with centers ζ_K and diameters $|\zeta_K|^{-S}$ by C_K, $K = 1, 2, \cdots$ where S is a sufficiently large positive number independent of K. Let $E(\delta)$ denote the set $[\bigcup_k C_k] \cap [|\arg z| < \delta < \pi/2]$. Finally, let $E^*(\delta)$ denote the circular projection (around the origin) of the set $E(\delta)$ on the real axis. Then we have the following result which is related to Theorem 1.

THEOREM 4. *If* $f(z)$ *is regular for* $\Re z > 0$ *and* $\limsup_{z \to i\eta} \log |f(z)| \leqq 0$, η *being a real number, and if* $\beta = \sup \log |f(z)| / \Re z < \infty$ *for* $\Re z > 0$, *then*

$$\lim_{z \to \infty} \log |f(z)| / \Re z = \beta \qquad \textit{for}\ |\arg z| < \delta < \pi/2$$

uniformly with the exception at most of a set $E(\delta)$ *such that*

$$\int_{E^*(\delta)} t^{s-1}dt < \infty. \tag{1.1}$$

The relations between the conclusions of Theorem 1 and 2 are clarified by the following fact

$$\int_{E^*(\delta)} t^{s-1}dt \geqq \int_{E^*(\delta)} d \log t.$$

By virtue of Boas' method [3], the following theorems are obtained.

THEOREM 5. *Let* $f(z)$ *be a regular function of exponential type in* $\Re z \geqq 0$. *If* $\int_1^{+\infty} t^{-2} \log|f(it)f(-it)|dt$ *exists and if* $\int_r^{2r} t^{-1} \log|f(\pm it)|dt = O(r)$, *then the conclusion of Theorem* 4 *holds.*

THEOREM 6. *If* $f(z)$ *is a regular function of exponential type in* $\Re z \geqq 0$ *and if* $\int_1^R t^{-2} \log|f(it)f(-it)|dt$ *is bounded above* $(1 < R < \infty)$, *then* $r^{-1}\log|f(re^{i\theta})|$ *is bounded for* $|\theta| < \pi/2$ *under the same exception as those of Theorem* 4.

2. **Carleman's theorem.** Let $u(z)$ be subharmonic for $\Re z \geqq 0$. Then by the Riesz theorem [9] there exists a generalized positive mass distribution $\mu(e)$ defined for bounded sets e of $\Re z > 0$ which are measurable (B) such that for $|z| < R$

$$(2.1)\qquad \begin{aligned} u(z) = {} & \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} u(Re^{i\phi})K_1(Re^{i\phi}, z)d\phi \\ & + \frac{1}{\pi}\int_{-R}^{+R} u(it)K_2(it, z)dt - \int_{|\zeta|<R} g_R(z, \zeta)d\mu(e_\zeta) \end{aligned}$$

where $K_1(Re^{i\phi}, z)$ and $K_2(it, z)$ denote the F. and R. Nevanlinna's kernel [8] defined respectively by

$$(2.2)\qquad \begin{aligned} & K_1(Re^{i\phi}, re^{i\theta}) \\ & = \frac{2Rr(R^2 - r^2)\cos\phi\cos\theta}{\{R^2 + r^2 - 2Rr\cos(\phi - \theta)\}\{R^2 + r^2 + 2Rr\cos(\phi + \theta)\}}, \end{aligned}$$

$$(2.3)\qquad \begin{aligned} & K_2(it, re^{i\theta}) \\ & = \left\{\frac{1}{t^2 + r^2 - 2tr\sin\theta} - \frac{R^2}{R^4 + t^2r^2 - 2R^2tr\sin\theta}\right\} r\cos\theta \end{aligned}$$

and

$$g_R(z, \zeta) = \log\left|\frac{(z + \bar\zeta)(R^2 - z\bar\zeta)}{(z - \zeta)(R^2 + z\zeta)}\right|.$$

Let $E(\rho, \epsilon)$ denote a set of e of $\{r \leqq |z| < \rho - \epsilon,\ \rho + \epsilon \leqq |z| < R,\ |\arg z| < \pi/2 - \epsilon\ (\epsilon > 0)\}$. By the Carleman's method [10], consider the integral $I = 1/2\pi i \int\int_{E(\rho,\epsilon)\ni\zeta} \log(z+\bar\zeta)(R^2 - z\bar\zeta)/(z-\zeta)(R^2+z\zeta)d\mu(e_\zeta) \cdot (1/z^2 + 1/\rho^2)dz$ taken along the contour of the domain $[|\arg z| < \pi/2,\ r_1 < |z| < \rho]$ $(r_1 < r)$ in the positive sense, starting from the point $z = -i\rho$ with a fixed determination of the logarithm. Then we have

$$\Re I = -\frac{1}{2\pi r_1}\int_{-\pi/2}^{+\pi/2}\int_{E(\rho,\epsilon)\ni\zeta} g_R(r_1e^{i\theta},\zeta)\cos\theta\left(1+\frac{r_1^2}{\rho^2}\right)d\mu(e_\zeta)d\theta$$
$$-\frac{1}{2\pi r_1}\int_{-\pi/2}^{+\pi/2}\int_{E(\rho,\epsilon)\ni\zeta}\arg\frac{(r_1e^{i\theta}+\bar{\zeta})(R^2-r_1e^{i\theta}\bar{\zeta})}{(r_1e^{i\theta}-\zeta)(R^2+r_1e^{i\theta}\zeta)}$$
$$\cdot\sin\theta\left(1-\frac{r_1^2}{\rho^2}\right)d\mu(e_\zeta)d\theta$$
$$+\frac{1}{\pi\rho}\int_{-\pi/2}^{+\pi/2}\int_{E(\rho,\epsilon)\ni\zeta} g_R(\rho e^{i\theta},\zeta)\cos\theta d\mu(e_\zeta)d\theta.$$

Letting $r_1\to 0$ for fixed r, we get

$$\Re I = -\int_{E(\rho,\epsilon)\ni\zeta}\left(\Re\frac{1}{\zeta}-\frac{\Re\zeta}{R^2}\right)d\mu(e_\zeta)$$
$$(2.4)\qquad +\frac{1}{\pi\rho}\int_{-\pi/2}^{+\pi/2}\int_{E(\rho,\epsilon)\ni\zeta} g_R(\rho e^{i\theta},\zeta)\cos\theta d\mu(e_\zeta)d\theta.$$

Again, integrating by parts and using the theorem of residues,

$$(2.5)\ \Re I = -\int_{E(\rho,\epsilon)\ni\zeta,|\zeta|<(\rho-\epsilon)}\rho^{-2}\Re\zeta d\mu(e_\zeta)+\int_{E(\rho,\epsilon)\ni\zeta,|\zeta|<(\rho-\epsilon)}\Re\frac{1}{\zeta}d\mu(e_\zeta).$$

Therefore from (2.4) and (2.5)

$$\frac{1}{\pi\rho}\int_{-\pi/2}^{+\pi/2}\int_{E(\rho,\epsilon)\ni\zeta} g_R(\rho e^{i\theta},\zeta)\cos\theta d\mu(e_\zeta)d\theta$$
$$=\int_{E(\rho,\epsilon)\ni\zeta,|\zeta|<(\rho-\epsilon)}\left\{2\Re\frac{1}{\zeta}-\left(\frac{1}{\rho^2}+\frac{1}{R^2}\right)\Re\zeta\right\}d\mu(e_\zeta)$$
$$+\int_{E(\rho,\epsilon)\ni\zeta,(\epsilon+\rho)\leq|\zeta|<R}\left(\Re\frac{1}{\zeta}-\frac{\Re\zeta}{R^2}\right)d\mu(e_\zeta).$$

From (2.1), $\int_{|\zeta|<R}\Re\zeta d\mu(e_\zeta)<\infty$. Therefore using the convergence theorem, we find as $\epsilon\to 0$ and $r\to 0$,

$$\frac{1}{\pi\rho}\int_{-\pi/2}^{+\pi/2}\int_{|\zeta|<R} g_R(\rho e^{i\theta},\zeta)\cos\theta d\mu(e_\zeta)d\theta$$
$$(2.6)\qquad =\int_{|\zeta|<\rho}\left\{2\Re\frac{1}{\zeta}-\left(\frac{1}{\rho^2}+\frac{1}{R^2}\right)\Re\zeta\right\}d\mu(e_\zeta)$$
$$+\int_{\rho\leq|\zeta|<R}\left(\Re\frac{1}{\zeta}-\frac{\Re\zeta}{R^2}\right)d\mu(e_\zeta).$$

Next, let $m(r)=\pi^{-1}\int_{-\pi/2}^{+\pi/2}u(re^{i\theta})\cos\theta d\theta$. Then

$$(2.7)\qquad \begin{aligned} &\frac{1}{\pi^2\rho}\int_{-\pi/2}^{+\pi/2}\int_{-\pi/2}^{+\pi/2}u(Re^{i\phi})K_1(Re^{i\phi},\rho e^{i\theta})\cos\theta d\phi d\theta \\ &\quad = \frac{1}{\pi^2}\int_{-\pi/2}^{+\pi/2}u(Re^{i\phi})\int_{-\pi/2}^{+\pi/2}\frac{\cos\theta}{\rho}K_1(Re^{i\phi},\rho e^{i\theta})d\theta d\phi = \frac{m(R)}{R}. \end{aligned}$$

Since $u(re^{i\theta})$ is L-integrable on intervals $[-\pi/2\leqq\theta<\pi/2]$ and $[-R\leqq r\leqq R, |\theta|=\pi/2]$, for $\epsilon>0$

$$\begin{aligned} &\lim_{\epsilon\to 0}\frac{1}{\pi^2\rho}\int_{-\pi/2+\epsilon}^{+\pi/2-\epsilon}\int_{-R}^{+R}u(it)K_2(it,\rho e^{i\theta})\cos\theta dt d\theta \\ (2.8)\quad &= \lim_{\epsilon\to 0}\frac{1}{\pi^2\rho}\int_{-R}^{+R}u(it)\int_{-\pi/2+\epsilon}^{+\pi/2-\epsilon}K_2(it,\rho e^{i\theta})\cos\theta d\theta dt \\ &= \frac{1}{2\pi}\left\{\frac{1}{\rho^2}\int_{|t|<\rho}u(it)dt+\int_{\rho\leq|t|<R}t^{-2}u(it)dt-\int_{|t|<R}R^{-2}u(it)dt\right\}. \end{aligned}$$

Hence, from (2.6), (2.7) and (2.8), we get the following theorem which is closely related to the Carleman's theorem [**10**].

THEOREM A. *If $u(z)$ is a subharmonic function for $\Re z\geqq 0$ with positive Radon measure $\mu(e)$ defined for bounded sets e of $\Re z>0$,*

$$\begin{aligned} \frac{m(\rho)}{\rho} = &-\int_{|\zeta|<\rho}\left\{2\Re\frac{1}{\zeta}-\left(\frac{1}{\rho^2}+\frac{1}{R^2}\right)\Re\zeta\right\}d\mu(e_\zeta) \\ (2.9)\quad &-\int_{\rho\leqq|\zeta|<R}\left(\Re\frac{1}{\zeta}-\frac{\Re\zeta}{R^2}\right)d\mu(e_\zeta) \\ &+\frac{1}{2\pi}\left\{\int_{|t|<\rho}\frac{u(it)}{\rho^2}dt+\int_{\rho\leq|t|<R}\frac{u(it)}{t^2}dt-\int_{|t|<R}\frac{u(it)}{R^2}dt\right\} \\ &+\frac{m(R)}{R}. \end{aligned}$$

By the second mean value theorem, we find for $0<M<M'<\rho<R'<R$

$$\begin{aligned} \nu(\rho,R) &= \int_{|t|<\rho}u(it)\left(\frac{1}{\rho^2}-\frac{1}{R^2}\right)dt+\int_{\rho\leq|t|<R}u(it)\left(\frac{1}{t^2}-\frac{1}{R^2}\right)dt \\ (2.10)\quad &= \int_{|t|<M}u(it)\left(\frac{1}{\rho^2}-\frac{1}{R^2}\right)dt+\frac{R^2-\rho^2}{R^2}\int_{M'<|t|<\rho}\frac{u(it)}{t^2}dt \\ &\quad +\frac{R^2-\rho^2}{R^2}\int_{\rho\leq|t|<R}\frac{u(it)}{t^2}dt. \end{aligned}$$

Hence, if the condition (c) is satisfied, then $\nu(\rho)=\limsup_{R\to\infty}\nu(\rho, R)<\infty$ and $\limsup_{\rho\to\infty}\nu(\rho)\leqq 0$. If $\liminf_{R\to\infty} R^{-1}m(R)=\mu<\infty$, we can select a sequence $\{R_n\}$ of R such that $R_n\to\infty$ as $n\to\infty$ and get $\lim_{n\to\infty} R_n^{-1}m(R_n)=\mu$. If, for such a sequence $\{R_n\}$, we write $\nu_1(\rho)=\lim_{n\to\infty}\nu(\rho, R_n)$, then $\nu_1(\rho)\leqq\nu(\rho)$. So that

$$\limsup_{\rho\to\infty}\nu_1(\rho)\leqq\limsup_{\rho\to\infty}\nu(\rho). \tag{2.11}$$

On the other hand, for all ρ and R such that $\rho<R$,

$$\begin{aligned} -\int_{|\zeta|<\rho}\left\{2\Re\frac{1}{\zeta}-\left(\frac{1}{\rho^2}+\frac{1}{R^2}\right)\Re\zeta\right\}d\mu(e_\zeta)& \\ -\int_{\rho\leqq|\zeta|<R}\left(\Re\frac{1}{\zeta}-\frac{\Re\zeta}{R^2}\right)d\mu(e_\zeta)&\leqq 0. \end{aligned} \tag{2.12}$$

Thus from (2.9), (2.11) and (2.12), $\limsup_{\rho\to\infty}\rho^{-1}m(\rho)\leqq\mu$. Hence $\lim_{\rho\to\infty}\rho^{-1}m(\rho)=\mu$. In the above argument we find that the case $\mu=-\infty$ does not occur. Therefore we have

COROLLARY. *If $u(z)$ is subharmonic for $\Re z\geqq 0$ and satisfies* (c), *then*

$$\lim_{R\to\infty}R^{-1}m(R)=\mu \qquad (\infty\geqq\mu>-\infty)$$

exists.

The case where $u(it)$ is bounded above is due (in a slightly sharper form) to Ahlfors [1] and Heins [5].

3. **Proofs of Theorems** 2 **and** 3. Let (a) and (b) be satisfied. Then by Theorem A, $\lim_{R\to\infty}\int_{|\zeta|<R}(\Re(1/\zeta)-\Re\zeta/R^2)d\mu(e_\zeta)$ exists. Therefore,

$$\int_{1<|\zeta|<\infty}\Re\frac{1}{\zeta}\,d\mu(e_\zeta)<\infty. \tag{3.1}$$

Accordingly,

$$\lim_{R\to\infty}\int_{|\zeta|<R}g_R(z,\zeta)d\mu(e_\zeta)=\int_{|\zeta|<\infty}\log\left|\frac{z+\bar\zeta}{z-\zeta}\right|d\mu(e_\zeta). \tag{3.2}$$

Moreover, by Corollary, $\lim_{R\to\infty}\int_{|t|<R}u(it)(1/t^2-1/R^2)dt$ exists. Hence, if we select N sufficiently large for an arbitrary $\epsilon>0$, then by (b), for $r_2>r_1>N$, we find

$$\left|\int_{r_1<|t|<r_2}t^{-2}u(it)dt\right|<\epsilon.$$

Using the following equality

$$\int_{r_1}^{r_2} t^{-2}u(\pm it)dt = \int_{r_1<|t|<r_2} t^{-2}u(it)dt - \int_{r_1}^{r_2} t^{-2}u(\mp it)dt$$

where the two upper signs or two lower signs are to be taken we find

$$\lim_{R\to\infty} \int_1^R t^{-2}u(\pm it)dt \text{ exists.} \tag{3.3}$$

Next we consider

$$\begin{aligned} I_1 &= \int_{-\pi/2}^{+\pi/2} u(Re^{i\phi})K_1(Re^{i\phi}, z)d\phi \\ &= \frac{2\Re z}{\pi R}\int_{-\pi/2}^{+\pi/2} u(Re^{i\phi}) \cos\phi\{1 + \eta(R, \phi)\}d\phi \end{aligned}$$

where $\eta(R, \phi)\to 0$ as $R\to\infty$.

$$\left|\int_{-\pi/2}^{+\pi/2} u(Re^{i\phi}) \cos\phi\eta(R, \phi)d\phi\right| \leqq \max_\phi |\eta(R, \phi)| \int_{-\pi/2}^{+\pi/2} (u^+ + u^-) \cos\phi d\phi.$$

Hence, if (a) and (b) are satisfied, by corollary,

$$\lim_{n\to\infty} \int_{-\pi/2}^{+\pi/2} \frac{u^+ + u^-}{R_n} \cos\phi d\phi < \infty$$

for a sequence $\{R_n\}$ of R such that $R_n\to\infty$ as $n\to\infty$. So that $\mu<\infty$ and

$$\lim_{n\to\infty} I_1 = 2\mu\Re z.$$

Thus by Theorem 1 and Boas' studies we find the conclusion of Theorem 2.

For the proof of Theorem 3, $\epsilon=-1$ is typical. By virtue of the Boas' method, we estimate

$$\begin{aligned} \int_{-R}^{+R} \frac{u(it)dt}{t^2 + r^2 - 2tr\sin\theta} &= \int_0^R \frac{u(it) + u(-it)}{t^2 + r^2 - 2tr\sin\theta} dt \\ &\quad - \int_0^R \frac{4tr\sin\theta u(-it)}{(t^2 + r^2)^2 - 4t^2r^2\sin^2\theta} dt = I_2 + I_3. \end{aligned} \tag{3.4}$$

If (a) and (c) are satisfied, we find that

$$\int_1^{+\infty} t^{-2}\{u(it) + u(-it)\}dt \text{ exists.}$$

Hence I_2 approaches a limit which is

$$I_4 = \int_0^\infty \frac{u(it) + u(-it)}{t^2 + r^2 - 2tr\sin\theta}\,dt. \tag{3.5}$$

So that for $|\theta| < \pi/2$, $\lim_{r\to\infty} I_4 = 0$. If (d) is satisfied, $\int_1^\infty t^{-3}u(-it)dt$ exists. Since $t^4/\{(t^2+r^2)^2 - 4t^2r^2\sin^2\theta\}$ increases if $|\theta| \leqq \pi/4$ or $\pi/2 > |\theta| > \pi/4$ and $t < r/(-\cos 2\theta)^{1/2}$, and decreases if $\pi/2 > |\theta| > \pi/4$ and $t > r/(-\cos 2\theta)^{1/2}$, I_3 approaches a limit which is

$$I_5 = \int_0^\infty \frac{4tr\sin\theta u(-it)}{(t^2 + r^2)^2 - 4t^2r^2\sin^2\theta}\,dt. \tag{3.6}$$

For the case $|\theta| \leqq \pi/4$, we write for $0 < r_1 < r < r_2 < \infty$

$$I_5 = \int_0^r + \int_r^\infty = \frac{r\sin\theta}{\cos^2\theta}\int_{r_1}^r t^{-3}u(-it)dt + 4r\sin\theta\int_{r_2}^\infty t^{-3}u(-it)dt.$$

Next, for the case $\pi/2 > |\theta| > \pi/4$, we have for $R_2 > R_1 > r > r' > 0$

$$I_5 = \int_0^r + \int_r^{r/(-\cos 2\theta)^{1/2}} + \int_{r/(-\cos 2\theta)^{1/2}}^\infty$$
$$= \frac{r\sin\theta}{\cos^2\theta}\int_{r'}^r t^{-3}u(-it)dt + \frac{r\sin\theta}{\sin^2 2\theta}\int_{R_1}^{R_2} t^{-3}u(-it)dt.$$

Therefore, if (d) and (e) are satisfied, $\lim_{r\to\infty} I_5 = 0$ for $|\theta| < \delta < \pi/2$.

Moreover we estimate

$$I_6 = \int_{-R}^{+R} \frac{R^2u(it)dt}{R^4 + t^2r^2 - 2R^2\,tr\sin\theta} = R^2\int_0^R \frac{u(it) + u(-it)}{R^4 + t^2r^2 - 2R^2\,tr\sin\theta}\,dt$$
$$-4R^4tr\sin\theta\int_0^R \frac{u(-it)dt}{(R^4 + t^2r^2)^2 - 4R^4t^2r^2\sin\theta}.$$

Both $t^2/\{R^4 - 2R^2tr\sin\theta + t^2r^2\}$ and $t^4/\{(R^4+t^2r^2)^2 - 4R^4t^2r^2\sin^2\theta\}$ increase for $r \leqq R$. Therefore we easily see $\lim_{R\to\infty} I_6 = 0$.

Thus we complete the proof of Theorem 3.

4. **Proof of Theorem** 4. By (2.1),

$$\int_{|\zeta|<1} \Re\zeta d\mu(e_\zeta) < \infty. \tag{4.1}$$

Let $D_K(\delta)$ denote a domain $[2^{K-2} \leqq |\zeta| < 2^{K+1}] \cap [|\arg \zeta| < \delta]$ $(0 < \delta < \pi/2)$, and put $A_K(\delta) = \int_{D_K(\delta) \ni \zeta} |\zeta|^{-1} d\mu(e_\zeta)$. Then from (3.1),

$$\sum_K A_K(\delta) < \infty. \tag{4.2}$$

Let $D_K^*(\delta^*)$ denote a domain $[2^{K-1} \leqq |z| < 2^K] \cap [|\arg z| < \delta^*]$ $(0 < \delta^* < \delta)$ and let

$$P(z) - \int_{|\zeta|<\infty} \log \left| \frac{z + \bar{\zeta}}{z - \zeta} \right| d\mu(\rho_\zeta). \tag{4.3}$$

We write

$$P(z) = P_1(z) + P_2(z) \tag{4.4}$$

where $P_1(z)$ and $P_2(z)$ denote respectively the potentials with the mass distribution on $D_K(\delta)$ and on the complement $\overline{D}_K(\delta)$ of $D_K(\delta)$ with respect to $\Re z > 0$. Let ϵ be an arbitrary positive number and let K_0 be a sufficiently large number. Then for all $K > K_0$, we obtain for $z \in D_K^*(\delta^*)$,

$$P_2(z)/\Re z < \epsilon. \tag{4.5}$$

Let η_K denote the number of the zero points of $f(z)$ in $D_K(\delta)$. Then

$$\eta_K \leqq 2^{K+1} A_K(\delta). \tag{4.6}$$

Let $\overline{C}_K$ denote the complement of C_K in $\Re z > 0$. Then for $z \in [(\bigcap_K \overline{C}_K) \cap D_K^*(\delta^*)]$ and $K > K_1$, K_1 a sufficiently large number,

$$\begin{aligned} \frac{P_1(z)}{\Re z} &\leqq \frac{\{K(s+1) + s + 2\} \log 2}{2^{K-1} \cos \delta^*} \int_{D_K(\delta) \ni \zeta} d\mu(e_\zeta) \\ &\leqq \frac{4\{K(s+1) + s + 2\} \log 2}{\cos \delta^*} A_K(\delta). \end{aligned} \tag{4.7}$$

On the other hand,

$$\lim_{K \to \infty} K A_K(\delta) = 0. \tag{4.8}$$

Choose K_0 sufficiently large for an arbitrary positive number ϵ. Then for all $K > K_0$ and $z \in [(\bigcap_K \overline{C}_K) \cap D_K^*(\delta^*)]$

$$P_1(z)/\Re z < \epsilon. \tag{4.9}$$

Concerning the total length of diameters of the exceptional circular disks, we get for $\zeta_K \in D_K(\delta)$

$$\sum_{i=1}^{\eta_K} |\zeta_i|^{-s} \leqq 2^{-s(K-2)} \eta_K \leqq 2^{-s(K-2)+K+1} A_K(\delta). \tag{4.10}$$

Hence

$$\sum_{K=K_1}^{\infty}\left\{\sum_{i=1}^{\eta_K}|\zeta_i|^{-s}2^{(K+1)(s-1)}\right\} \leqq 8^s \sum_{K=K_1}^{\infty} A_K(\delta) < \infty .$$

Accordingly, for the exceptional set, we get

$$\int_{E^*(\delta)} t^{s-1}dt < \infty . \tag{4.11}$$

Thus the proof is completed.

5. REMARK. In the proofs of Theorem 2 and 3, we find the following result: If $u(z)$ is subharmonic for $\Re z \geqq 0$ and satisfies the condition (a), then the conditions (b) and (c) are equivalent to the conditions (f) $\int_1^\infty t^{-2}u(\pm it)dt$ exist and (g) $\int_1^\infty t^{-2}\{u(it)+u(-it)\}dt$ exists respectively.

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