ON THE HANKEL J-, Y- AND H-TRANSFORMS

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Consider the Hankel transforms

$$g_j(\xi) = \int_0^\infty xJ_\nu(\xi x)G(x)dx \tag{1}$$

and

$$g_y(\xi) = \int_0^\infty xY_\nu(\xi x)G(x)dx \tag{2}$$

of a function $G(x)$. It is interesting to note that $g_j(\xi)$ and $g_y(\xi)$ are related by a formula which may be written as a Hilbert transform.

Assume that $-1/2<\nu<1/2$ and that p and q are positive real numbers. Then, consider the integral $\oint z^\nu H_\nu^{(1)}(qz)(p-z)^{-1}dz$ taken around a contour consisting of (i) a large semicircle (above the real axis) with centre the origin 0 and radius R, (ii) a small semicircle (above the real axis) with centre 0 and radius ϵ, (iii) a small semicircle C (above the real axis) with centre the point $z=p$ and radius η, and (iv) the parts of the real axis joining the ends of these semicircles. Using well known properties of the Bessel functions concerned, and in particular Watson [2, p. 75(5)] we easily obtain[1]

$$\left[\int_0^{p-\eta} + \int_{p+\eta}^\infty\right] \frac{x^\nu H_\nu^{(1)}(qx)}{p-x}\,dx \\ - \int_0^\infty \frac{x^\nu H_\nu^{(2)}(qx)}{p+x}\,dx + \int_C \frac{z^\nu H_\nu^{(1)}(qz)}{p-z}\,dz = 0.$$

In order to simplify the notation we introduce $U(x, p, \eta)$, defined so that $U(x, p, \eta)=0$ for $p-\eta<x<p+\eta$ and $=1$ otherwise.

Now, extracting the real and imaginary parts of the last equation we obtain

$$\int_0^\infty x^\nu J_\nu(qx)\left[\frac{U(x, p, \eta)}{p-x} - \frac{1}{p+x}\right]dx = -\operatorname{Re}\int_C \frac{z^\nu H_\nu^{(1)}(qz)}{p-z}\,dz \tag{3a}$$

which leads to

Received by the editors February 24, 1958 and, in revised form, March 21, 1958.

[1] In this equation and where it occurs later the integral involving the semicircle C is taken from left to right. Also in equation (3b) and other equations the sign $⨍$ denotes the Cauchy principal value of the integral.

$$\text{(3b)} \qquad \mathop{\int\!\!\!\!\!\!-}\nolimits_0^\infty x^\nu J_\nu(qx)\left[\frac{1}{p-x}-\frac{1}{p+x}\right]dx = \pi p^\nu Y_\nu(qp),$$

and

$$\text{(4a)} \quad \int_0^\infty x^\nu Y_\nu(qx)\left[\frac{U(x,\,p,\,\eta)}{p-x}+\frac{1}{p+x}\right]dx = -\operatorname{Im}\int_C \frac{z^\nu H_\nu^{(1)}(qz)}{p-z}\,dz$$

which leads to

$$\text{(4b)} \qquad \mathop{\int\!\!\!\!\!\!-}\nolimits_0^\infty x^\nu Y_\nu(qx)\left[\frac{1}{p-x}+\frac{1}{p+x}\right]dx = -\pi p^\nu J_\nu(qp).$$

Then assuming that η is sufficiently small ($<p/2$ say) and that $x^{1/2}G(x)$ belongs to $L^1(0,\,\infty)$,

$$\begin{aligned}
&\int_0^\infty \left[\frac{U(\xi,p,\eta)}{p-\xi}-\frac{1}{p+\xi}\right]\xi^\nu g_j(\xi)d\xi \\
\text{(5)} \qquad &= \int_0^\infty\int_0^\infty \left[\frac{U(\xi,p,\eta)}{p-\xi}-\frac{1}{p+\xi}\right]\xi^\nu x G(x)J_\nu(\xi x)dxd\xi \\
&= \int_0^\infty x^{1/2}G(x)dx\int_0^\infty \left[\frac{U(\xi,p,\eta)}{p-\xi}-\frac{1}{p+\xi}\right]\xi^\nu x^{1/2}J_\nu(\xi x)d\xi
\end{aligned}$$

(by absolute convergence, since $(\xi x)^{1/2}J_\nu(\xi x)$ is bounded for all positive real ξx)

$$= -\int_0^\infty x^{1/2}G(x)dx \operatorname{Re}\int_C x^{1/2}z^\nu H_\nu^{(1)}(xz)(p-z)^{-1}dz$$

(by Equation (3a)).

Now $z^{1/2}H_\nu^{(1)}(z)$ is bounded for all z with $\operatorname{Im} z \geqq 0$. Let $|z^{1/2}H_\nu^{(1)}(z)| < M$. So

$$\begin{aligned}
\left|\int_C x^{1/2}z^\nu H_\nu^{(1)}(xz)(p-z)^{-1}dz\right| &\leqq \int_C |z|^{\nu-1/2}M\eta^{-1}|dz| \\
&\leqq M\pi(p-\eta)^{\nu-1/2} \\
&< M\pi(p/2)^{\nu-1/2}.
\end{aligned}$$

This shows that we may use bounded convergence to take the limit $\eta\to 0$ in Equation (5). Thus

$$\int_0^\infty \left[\frac{1}{p-\xi} - \frac{1}{p+\xi}\right] \xi^\nu g_j(\xi) d\xi = \int_0^\infty xG(x)\pi p^\nu Y_\nu(px)dx \tag{6}$$
$$= \pi p^\nu g_y(p).$$

We may proceed similarly with an integral involving $g_y(\xi)$. The results are summarized by

THEOREM 1. *If* (i) $x^{1/2}G(x)$ *belongs to* $L^1(0, \infty)$, (ii) $-1/2<\nu<1/2$ *and* (iii) $g_j(\xi)$ *and* $g_y(\xi)$ *are defined by Equations* (1) *and* (2) *respectively then*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\xi|^\nu g_j(|\xi|)\operatorname{sgn}\xi}{\xi-p}\, d\xi = -|p|^\nu g_y(|p|), \tag{7}$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\xi|^\nu g_y(|\xi|)}{\xi-p}\, d\xi = |p|^\nu g_j(|p|)\operatorname{sgn} p \tag{8}$$

where both integrals are to be understood as Cauchy principal value integrals.

We terminate the first part of this note by observing that results (3b), (4b) (7) and (8) may, with an easy change of variable, be written as

$$\frac{1}{\pi}\int_0^\infty \frac{x^{\nu/2}J_\nu(qx^{1/2})}{p-x}\, dx = p^{\nu/2}Y_\nu(qp^{1/2}),$$
$$\frac{1}{\pi}\int_0^\infty \frac{x^{(\nu-1)/2}Y_\nu(qx^{1/2})}{p-x}\, dx = -p^{(\nu-1)/2}J_\nu(qp^{1/2}), .$$
$$\frac{1}{\pi}\int_0^\infty \frac{x^{\nu/2}g_j(x^{1/2})}{p-x}\, dx = p^{\nu/2}g_y(p^{1/2}),$$
$$\frac{1}{\pi}\int_0^\infty \frac{x^{(\nu-1)/2}g_y(x^{1/2})}{p-x}\, dx = -p^{(\nu-1)/2}g_j(p^{1/2}).$$

The first of these has been recorded in Erdélyi [**1**, p. 256(27)] and Equation (4b) above appears in [**1**, p. 101 (22)] in a modified form.

By considering the integral $\oint z^{-\nu}[J_\nu(qz)+i\boldsymbol{H}_\nu(qz)](z-p)^{-1}dz$ taken around the contour used above it is easy to derive

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|x|^{-\nu}J_\nu(q|x|)}{x-p}\, dx = -|p|^\nu \boldsymbol{H}_\nu(q|p|)\operatorname{sgn} p \tag{9}$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|x|^{-\nu}\mathbf{H}_\nu(q|x|)\operatorname{sgn} x}{x-p}\,dx = |p|^{\nu}J_\nu(q|p|) \tag{10}$$

where $\mathbf{H}_\nu(z)$ is Struve's function of order ν [1, p. 255 (15)].

If we then write the $\mathbf{H}$-transform as

$$g_h(\xi) = \int_0^\infty x\mathbf{H}_\nu(\xi x)G(x)dx \tag{11}$$

we may prove, by a method analogous to that used in Theorem 1, that the following theorem holds:—

THEOREM 2. *If* (i) $x^{1/2}G(x)$ *belongs to* $L^1(0, \infty)$, (ii) $-1/2<\nu<1/2$ (iii) $g_j(\xi)$ *and* $g_h(\xi)$ *are defined by Equations* (1) *and* (11) *above, then when* p *is real*

$$\frac{1}{\pi}\int_{-\infty}^{-\infty}\frac{|x|^{-\nu}g_j(|x|)}{x-p}\,dx = -|p|^{-\nu}g_h(|p|)\operatorname{sgn} p, \tag{12}$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|x|^{-\nu}g_h(|x|)(\operatorname{sgn} x)}{x-p}\,dx = |p|^{-\nu}g_j(|p|). \tag{13}$$

REFERENCES

1. A. Erdelyi and others, *Tables of integral transforms*, vol. 2, New York, McGraw-Hill, 1954.

2. G. N. Watson, *Theory of Bessel functions*, Cambridge, Cambridge University Press, 1958.

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