ANOTHER SUBDIVISION WHICH CAN NOT BE SHELLED

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1. **Introduction.** Let B be a 3-cell, and consider a subdivision of B into 3-cells $B_1, \cdots, B_k$. Let K_i denote the union of the B_j other than B_i. I will say that the subdivision *can not be shelled* in case, for each i, K_i is not a 3-cell. Several examples of such subdivisions have been constructed, see [**3**]; the most remarkable is M. E. Rudin's [**1**], in which B and all the B_i are tetrahedra (k is 41). The example given here is quite simple, and the value of k is obviously minimal! However, it is not a *simplicial* decomposition.

THEOREM 1. *There exists a subdivision of the 3-cell into three pieces which can not be shelled.*

2. **The example.** Introduce Cartesian coordinates x_1, x_2, x_2. Let B be $x_1^2+x_2^2+x_3^2 \leqq M^2$, where M is a sufficiently large positive number. Let T_i be the line $x_{i+1}=1$, $x_{i+2}=-1$; these subscripts are modulo 3. Let B_i consist of those points p of B such that

$$d(p,\ T_i) = \min_{j=1,2,3} d(p,\ T_j).$$

(Here $d(\ ,\ )$ is Euclidean distance between closed sets.)

B_1 is a 3-cell by the following argument. It is permissible to ignore the small caps where $x_1^2 \geqq M^2-1$. Consider any other cross-section of B_1 by a plane $x_1=$constant. It is star-shaped about the point where T_1 cuts it; for going from $p=(x_1, x_2, x_3) \in B_1$ to $p_\lambda=(x_1, 1+\lambda(x_2-1), -1+\lambda(x_3+1))$ with $0 \leqq \lambda < 1$ reduces the distance to T_1 by $d(p, p_\lambda)$ and can not reduce the distance to T_2 or T_3 by more. It contains a centered disk, namely $(x_2-1)^2+(x_3+1)^2 \leqq 1$. It is closed. This gives an obvious method for deforming B_1 onto a cylinder.

$K_1 = B_2 \cup B_3$ is not a cell, for its fundamental group does not vanish. Namely, consider $x_1=0$, $x_2^2+x_3^2=M^2$. If p is on this circle, $d(p, T_1) \geqq M-2^{1/2}$, while at least one of $d(p, T_2)$ and $d(p, T_3)$ is $\leqq (M^2/2+1)^{1/2}$. Since M is large, $p \in K_1$. This circle is not contractible in K_1, for K_1 has no points in $(x_2-1)^2+(x_3+1)^2 < 1$.

Because of the evident symmetry between the B_i, this proves Theorem 1.

3. **Generalization.**

THEOREM 2. *For $k \geqq 3$ there exists a subdivision of the 3-cell into k pieces which can not be shelled.*

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This result, which was proposed by O. G. Harrold, is clearly a consequence of the 3-dimensional case of the following theorem. (If the conclusion of Theorem 3 was simply that the subdivision could not be shelled, it could conversely be deduced from Theorem 2.)

THEOREM 3. *For $k \geqq n \geqq 3$ there exists a subdivision of the n-cell into k pieces such that the union of any $k-1$ of the pieces has nontrivial $n-2$th homotopy group.*

The idea of the construction is the same as for Theorem 1. Choose k vectors, the first n of which are the coordinate unit vectors, and no n of which are linearly dependent. Choose any nonintersecting lines $T_1, \cdots, T_k$ in these respective directions, such that each $d(T_i, T_j)$ is at least 2. There exists $D>0$ such that each T_i passes within D of the origin. Imitate the definitions above of M, B and the B_i. The proof that B_i is a cell is not much affected.

The conclusion of the theorem concerns $\pi_{n-2}(K_i)$. We may choose Cartesian coordinates $y_1, \cdots, y_n$ such that T_i is the y_1-axis, and such that the $n-2$-sphere

$$y_1 = 0,\ y_2^2 + \cdots + y_n^2 = (M - D)^2$$

lies in B. Then this sphere lies in K_i. (This is proved separately for $i \leqq n$ and $i > n$.) It is not contractible in K_i, for K_i has no points in $y_2^2 + \cdots + y_n^2 < 1$.

4. **Another version of the example.** Let C_{1i} (for $i=1, 2, 3$) be the set of points with cylindrical coordinates

$$0 \leqq z \leqq 1, \qquad 0 \leqq r \leqq 1,$$

$$\frac{2\pi}{3}(i - 1) \leqq \theta - \frac{\pi}{2} r \leqq \frac{2\pi}{3} i.$$

Similarly define C_{2i} by

$$1 \leqq z \leqq 2, \qquad 0 \leqq r \leqq 1,$$

$$\frac{2\pi}{3}(i - 1) \leqq \theta + \frac{\pi}{2} r \leqq \frac{2\pi}{3} i.$$

Now if $B_i = C_{1i} \cup C_{2i}$ and $B = B_1 \cup B_2 \cup B_3$, a figure is obtained which is homeomorphic to the one so labeled in §2. The two versions yield Theorem 1 equally easily, but this one leads to a different generalization.

THEOREM 4. *For any* $\nu=1, 2, 3, \cdots$, *there exists a subdivision of the 3-cell into three pieces such that omitting any of the pieces leaves a set whose fundamental group is the free group on* ν *generators.*

In fact, define $C_{2\mu+1,i}$ as the result of translating C_{1i} along the z-axis a distance 2μ; $C_{2\mu+2,i}$ similarly in terms of C_{2i}. If $B_i = C_{1i} \cup \cdots \cup C_{\nu+1,i}$ and $B = B_1 \cup B_2 \cup B_3$, the resulting figure may be shown to provide the example for Theorem 4.

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