ANOTHER SUBDIVISION WHICH CAN NOT BE SHELLED

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1. Introduction. Let $B$ be a 3-cell, and consider a subdivision of $B$ into 3-cells $B_1, \cdots, B_k$. Let $K_i$ denote the union of the $B_j$ other than $B_i$. I will say that the subdivision can not be shelled in case, for each $i$, $K_i$ is not a 3-cell. Several examples of such subdivisions have been constructed, see [3]; the most remarkable is M. E. Rudin's [1], in which $B$ and all the $B_i$ are tetrahedra ($k$ is 41). The example given here is quite simple, and the value of $k$ is obviously minimal! However, it is not a simplicial decomposition.

**Theorem 1.** There exists a subdivision of the 3-cell into three pieces which can not be shelled.

2. The example. Introduce Cartesian coordinates $x_1, x_2, x_3$. Let $B$ be $x_1^2 + x_2^2 + x_3^2 \leq M^2$, where $M$ is a sufficiently large positive number. Let $T_i$ be the line $x_{i+1} = 1, x_{i+2} = -1$; these subscripts are modulo 3. Let $B_i$ consist of those points $p$ of $B$ such that

$$d(p, T_i) = \min_{j=1,2,3} d(p, T_j).$$

(Here $d(\cdot, \cdot)$ is Euclidean distance between closed sets.)

$B_1$ is a 3-cell by the following argument. It is permissible to ignore the small caps where $x_1^2 \geq M^2 - 1$. Consider any other cross-section of $B_1$ by a plane $x_1 = \text{constant}$. It is star-shaped about the point where $T_1$ cuts it; for going from $p = (x_1, x_2, x_3) \in B_1$ to $p_\lambda = (x_1, 1 + \lambda(x_2 - 1), -1 + \lambda(x_3 + 1))$ with $0 \leq \lambda < 1$ reduces the distance to $T_1$ by $d(p, p_\lambda)$ and can not reduce the distance to $T_2$ or $T_3$ by more. It contains a centered disk, namely $(x_2 - 1)^2 + (x_3 + 1)^2 \leq 1$. It is closed. This gives an obvious method for deforming $B_1$ onto a cylinder.

$K_1 = B_2 \cup B_3$ is not a cell, for its fundamental group does not vanish. Namely, consider $x_1 = 0, x_2^2 + x_3^2 = M^2$. If $p$ is on this circle, $d(p, T_1) \geq M - 2^{1/2}$, while at least one of $d(p, T_2)$ and $d(p, T_3)$ is $\leq (M^2/2 + 1)^{1/2}$. Since $M$ is large, $p \in K_1$. This circle is not contractible in $K_1$, for $K_1$ has no points in $(x_2 - 1)^2 + (x_3 + 1)^2 < 1$.

Because of the evident symmetry between the $B_i$, this proves Theorem 1.

3. Generalization.

**Theorem 2.** For $k \geq 3$ there exists a subdivision of the 3-cell into $k$ pieces which can not be shelled.
This result, which was proposed by O. G. Harrold, is clearly a consequence of the 3-dimensional case of the following theorem. (If the conclusion of Theorem 3 was simply that the subdivision could not be shelled, it could conversely be deduced from Theorem 2.)

**Theorem 3.** For \( k \geq n \geq 3 \) there exists a subdivision of the \( n \)-cell into \( k \) pieces such that the union of any \( k - 1 \) of the pieces has nontrivial \( n - 2 \)th homotopy group.

The idea of the construction is the same as for Theorem 1. Choose \( k \) vectors, the first \( n \) of which are the coordinate unit vectors, and no \( n \) of which are linearly dependent. Choose any nonintersecting lines \( T_1, \ldots, T_k \) in these respective directions, such that each \( d(T_i, T_j) \) is at least 2. There exists \( D > 0 \) such that each \( T_i \) passes within \( D \) of the origin. Imitate the definitions above of \( M, B \) and the \( B_i \). The proof that \( B_i \) is a cell is not much affected.

The conclusion of the theorem concerns \( \pi_{n-2}(K_i) \). We may choose Cartesian coordinates \( y_1, \ldots, y_n \) such that \( T_i \) is the \( y_1 \)-axis, and such that the \( n - 2 \)-sphere

\[
y_1 = 0, \ y_2 + \cdots + y_n^2 = (M - D)^2
\]

lies in \( B \). Then this sphere lies in \( K_i \). (This is proved separately for \( i \leq n \) and \( i > n \).) It is not contractible in \( K_i \), for \( K_i \) has no points in \( y_2^2 + \cdots + y_n^2 < 1 \).

4. **Another version of the example.** Let \( C_{1i} \) (for \( i = 1, 2, 3 \)) be the set of points with cylindrical coordinates

\[
0 \leq z \leq 1, \quad 0 \leq r \leq 1,
\]

\[
\frac{2\pi}{3} (i - 1) \leq \theta - \frac{\pi}{2} r \leq \frac{2\pi}{i}.
\]

Similarly define \( C_{2i} \) by

\[
1 \leq z \leq 2, \quad 0 \leq r \leq 1,
\]

\[
\frac{2\pi}{3} (i - 1) \leq \theta + \frac{\pi}{2} r \leq \frac{2\pi}{i}.
\]

Now if \( B_i = C_{1i} \cup C_{2i} \) and \( B = B_1 \cup B_2 \cup B_3 \), a figure is obtained which is homeomorphic to the one so labeled in §2. The two versions yield Theorem 1 equally easily, but this one leads to a different generalization.
Theorem 4. For any \( v = 1, 2, 3, \ldots \), there exists a subdivision of the 3-cell into three pieces such that omitting any of the pieces leaves a set whose fundamental group is the free group on \( v \) generators.

In fact, define \( C_{2^{\mu}+1, i} \) as the result of translating \( C_{1i} \) along the \( z \)-axis a distance \( 2\mu \); \( C_{2^{\mu+2}, i} \) similarly in terms of \( C_{2i} \). If \( B_i = C_{1i} \cup \cdots \cup C_{r+1, i} \) and \( B = B_1 \cup B_2 \cup B_3 \), the resulting figure may be shown to provide the example for Theorem 4.

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Bibliography


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