

ORDERED COMMUTATIVE SEMIGROUPS OF THE SECOND KIND¹

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By an *ordered commutative semigroup of the first kind* (abbreviated "o.c.s.I") we mean a system $S(\circ, <)$ consisting of a set S endowed with a binary operation \circ and a binary relation $<$ such that the following axioms are satisfied.

I. S is a commutative semigroup with respect to \circ , i.e. the associative law, $a \circ (b \circ c) = (a \circ b) \circ c$, and the commutative law, $a \circ b = b \circ a$, hold (a, b, c in S).

II. S is totally (=linearly=simply) ordered by $<$.

III. If a and b are elements of S such that $a < b$, then $a \circ c \leq b \circ c$ for all c in S .

Although this notion is very extensive, it does not include the ordered multiplicative system of all real numbers, in which multiplication by a negative number inverts the direction of any inequality. Let $S(\circ, <)$ satisfy just I and II above. An element c of S is called a *conserver* if $a < b$ (a, b in S) implies $c \circ a \leq c \circ b$, and an *inverter* if $a < b$ implies $c \circ a \geq c \circ b$. $S(\circ, <)$ will be called an *ordered commutative semigroup of the second kind* (abbreviated "o.c.s. II") if it satisfies I, II, and IV below.

IV. Every element of S is either a conserver or an inverter, or both.

Systems of this nature (in fact more general systems) have been investigated by Dov Tamari [2; 3]. His definition of conserver (inverter) is more restrictive: $a < b$ implies $c \circ a < c \circ b$ ($c \circ a > c \circ b$). If c is a conserver or inverter in his sense, c is cancellable ($c \circ a = c \circ b$ implies $a = b$). I have taken the liberty of relaxing the definition so as to apply to noncancellable elements as well; the term "strict conserver (inverter)" may be used for his concept.

The main objective of the first part of the present paper is to show that the set P of conservers of S and the set Q of inverters of S are convex. (A subset A of S is *convex* if $a \in A$, $a' \in A$, and $a < x < a'$ imply $x \in A$.)

An ultimate objective is to construct all o.c.s.II's from o.c.s.I's. In the second part of the paper we give such a construction for a fairly restricted class of such semigroups. This was suggested by recent work of Haskell Cohen and L. I. Wade [1].

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NOTATION. We shall write ab instead of $a \circ b$. If A and B are subsets of S , then (1) AB denotes the set of all ab with a in A , b in B ; (2) $A < B$ means $a < b$ for all a in A , b in B ; (3) $A \setminus B$ means the set of elements of A not in B .

1. **Location of conservers and inverters.** Throughout this section, S will denote an o.c.s.II. P and Q will denote the sets of conservers and inverters, respectively, of S . The elements of $N = P \cap Q$ will be called *null elements*. Evidently $P^2 \subseteq P$, $Q^2 \subseteq Q$, $PQ \subseteq Q$. In particular, P is a subsemigroup of S , and is an o.c.s.I. N is an ideal of S , i.e. $SN \subseteq N$, if it is not empty.

If S contains a zero element 0 , i.e. $a0 = 0a = 0$ for all a in S , we shall say that an element a of S is *positive* if $a > 0$ and *negative* if $a < 0$. We say that a and b *have the same sign* if they are both positive or both negative, and *have opposite sign* if one is positive and the other negative.

LEMMA 1. *Let S have a zero element 0 . Let a and b be elements of S . If a and b are both conservers or both inverters, but have opposite sign, then $ab = 0$. If a is a conserver and b an inverter, or vice-versa, and they have the same sign, then $ab = 0$.*

PROOF. Suppose $a < 0 < b$. If a and b are both conservers, $a < 0$ implies $ab \leq 0b = 0$, and $0 < b$ implies $0 = a0 \leq ab$, whence $ab = 0$. If a and b are both inverters, $a < 0$ implies $ab \geq 0b = 0$, and $0 < b$ implies $0 = a0 \geq ab$, and again we conclude $ab = 0$.

Suppose a is a conserver and b an inverter, and $a < 0$, $b < 0$. $a < 0$ implies $ab \geq 0$ since b is an inverter. $b < 0$ implies $ab \leq 0$ since a is a conserver. Hence $ab = 0$. The case $a > 0$, $b > 0$ is similar.

An element a of S is a null if and only if $ab = ac$ for all b, c in S . For if, say, $b < c$, then $ab \leq ac$ because a is a conserver, and $ab \geq ac$ because a is an inverter. From this we see that a is a null if and only if $ab = a^2$ for all b in S , i.e. $aS = \{a^2\}$.

LEMMA 2. *S contains a null element if and only if S contains a zero element 0 . An element a of S is a null if and only if $aS = \{0\}$. The set N of null elements of S is a convex ideal containing 0 .*

PROOF. Let a be a null element of S . Then a^2 is a zero element of S . For if b is any element of S , $a^2b = a(ab) = a^2$. Conversely, if S contains a zero element 0 , then 0 is evidently a null element. Since the zero element is unique, we have shown that $a^2 = 0$ for any null a . Thus $aS = \{a^2\} = \{0\}$ for any null a , and conversely if $aS = \{0\}$ then a is clearly null.

We have noted above that N is an ideal. To show that it is convex, let $a \in N$, $b \in N$, $c \in S$, and $a < c < b$. Let $x \in S$. Then $0 = ax \leq cx \leq bx = 0$ if x is a conserver, and $0 = ax \geq cx \geq bx = 0$ if x is an inverter. In either event, $cx = 0$. Hence $cS = \{0\}$ and $c \in N$.

REMARK 1. The above proof of the convexity of N carries over to the following: *the set of annihilators of any element or subset of S is convex.*

LEMMA 3. *If N , $P \setminus N$, and $Q \setminus N$ are all nonempty, then $P \setminus N < N < Q \setminus N$ or $Q \setminus N < N < P \setminus N$.*

PROOF. Assume first that S contains arbitrarily large conservers. Then S cannot contain a positive non-null inverter. For suppose a is an inverter, and $a > 0$. We proceed to show that a is null.

Let $x > 0$. By hypothesis there exists a conserver $b \geq x$. $ab = 0$ by Lemma 1, and hence $ax = 0$, from $0 < x \leq b$ and Remark 1. Let $x < 0$. If x is an inverter, then $ax = 0$ by Lemma 1. If x is a conserver, let c be a conserver $\geq a$; c exists, by assumption. Then $cx = 0$ by Lemma 1, and hence $ax = 0$ from $0 < a \leq c$ and Remark 1.

Similarly we show: if S contains arbitrarily large inverters, then S cannot contain a positive non-null conserver; if S contains arbitrarily small conservers (inverters), then S cannot contain a negative non-null inverter (conserver). From these the lemma follows at once.

LEMMA 4. *If S does not contain a zero element, then $P < Q$ or $Q < P$.*

PROOF. Since $S = P \cup Q$ and $P \cap Q$ is empty, by Lemma 2, it suffices to show that P and Q are convex.

Suppose, by way of contradiction, that $a < b < c$ with a, c in P and b in Q . Since b is an inverter, $a < b < c$ implies $ab \geq b^2 \geq bc$. Since a is a conserver, $b < c$ implies $ab \leq ac$. Since c is a conserver, $a < b$ implies $ac \leq bc$. Hence $bc \leq b^2 \leq ab \leq ac \leq bc$, whence $bc = b^2$. But $bc \in QP \subseteq Q$, $b^2 \in Q^2 \subseteq P$, contrary to the fact that $P \cap Q$ is empty. The proof of the convexity of Q is similar.

We combine the foregoing lemmas into the following theorem.

THEOREM 1. *Let S be an ordered commutative semigroup of the second kind. Let P be the set of conservers, Q the set of inverters of S , and $N = P \cap Q$ the set of nulls of S . If N is empty, then $P < Q$ or $Q < P$. If N is not empty, then S has a zero element 0 , $a \in N$ if and only if $aS = \{0\}$, and N is a convex ideal of S . If $P \setminus N$ and $Q \setminus N$ are not empty then N lies between them; in particular, the sets P , Q , N , $P \setminus N$ and $Q \setminus N$ are all convex.*

REMARK 2. If $Q \setminus N$ is empty then S is of the first kind. $P \setminus N$ need

not be convex in this case. In fact we may take any two o.c.s.I's, S_1 and S_2 , containing zero elements 0_1 and 0_2 , respectively, with 0_1 the greatest element of S_1 and 0_2 the least element of S_2 , and let S be their union, with 0_1 and 0_2 identified ($=0$, say). Order S so as to preserve the given order in S_1 and S_2 as already given, and make $S_1 < S_2 \setminus \{0\}$. Define products within S_1 and S_2 as already given, and for $a_1 \in S_1, a_2 \in S_2$, define $a_1 a_2 = a_2 a_1 = 0$. S becomes thereby an o.c.s.I. From Lemma 1 it follows that any o.c.s.I with interior zero element can be obtained in this fashion from its subsemigroups S_1 of all non-negative elements and S_2 of all non-positive elements.

REMARK 3. If $P \setminus N$ is empty, $Q \setminus N$ need not be convex. For example, let $S = \{a_1 < a_2 < c < 0 < d < b_1 < b_2\}$ with all products $= 0$ except $a_i a_j = d$ and $b_i b_j = c$ (all $i, j = 1, 2$). We note that if $P \setminus N$ is empty, then $S^2 \subseteq N$, and so $S^3 = \{0\}$. For $S = Q$, whence $S^2 = Q^2 \subseteq P = P \cap Q = N$.

2. **Construction of a certain class of o.c.s.II's.** Let S and S' be o.c.s.II's. By a *homomorphism* of S into S' we mean a mapping f of S into S' satisfying (i) $f(ab) = f(a)f(b)$ for all a, b in S , and (ii) if $a < b$ then $f(a) \leq f(b)$. By a *congruence relation* ρ in S we mean an equivalence relation such that $a\rho b$ implies $ac\rho bc$ for all c in S . ρ is called *convex* if each congruence class is convex. Let $[a]$ denote the congruence class to which a belongs (a in S). The *factor semigroup* S/ρ consists of all the classes $[a]$, with product defined by $[a] \cdot [b] = [ab]$. If ρ is convex, we can order S/ρ by defining $[a] < [b]$ if $[a] \neq [b]$ and $a < b$. S/ρ is then an o.c.s.II, and the *canonical mapping* $a \rightarrow [a]$ is a homomorphism of S onto S/ρ . If f is a homomorphism of S onto S' , and we define $a\rho b$ if and only if $f(a) = f(b)$, then ρ is a convex congruence relation in S , and $[a] \rightarrow f(a)$ is an isomorphism of S/ρ onto S' .

An example of a convex congruence relation which we shall meet later is $\tau(k)$, where k is a fixed element of S , defined as follows: $a\tau(k)b$ if and only if $ka = kb$.

We proceed now to investigate the class of o.c.s.II's S satisfying the following conditions:

- (a) S is not of the first kind.
- (b) S contains a zero element 0 , not an endpoint.
- (c) S has endpoints $\delta < u$, and u is the identity element of S (i.e. $ua = au = a$, all a in S).
- (d) $L \subseteq \delta R$, where $L = [\delta, 0]$ and $R = [0, u]$.

By (a), S contains a non-null inverter. S also contains the non-null conserver u . Since S contains an identity element, N must consist of 0 alone. By Theorem 1, 0 lies between the set $P \setminus N$ of non-null conservers and the set $Q \setminus N$ of non-null inverters. Since $u \in P \setminus N$, it

follows that $R = P$ and $L = Q$. In particular, $\delta \in Q$, and so $\delta R \subseteq QP \subseteq Q = L$. (d) requires that δR be all of L .

The above notation is that of Cohen and Wade [1]. They are concerned with the case where S is a real interval, $x \circ y$ is a continuous function of the two variables x and y , and product \circ is not necessarily commutative. (d) holds because the continuous mapping $x \rightarrow \delta x$ maps the connected set R onto the connected set δR containing 0 and δ , hence containing L . There are two main cases in [1], depending on whether $L^2 \subseteq L$ or $L^2 \subseteq R$. We are concerned only with the latter. Theorem 2 below may be regarded as an algebraic extract from Cohen and Wade's determination of S in this case. The present treatment will be self-contained.

R is a subsemigroup of S , and is an o.c.s.I, with the zero element at the lower end and the identity element at the upper end. We take the point of view that R is known, and our objective is to construct S in all possible ways from R .

First assume that S is given. Then $k = \delta^2 \in R$. Let $\phi(x) = \delta x$ ($x \in R$). By (d), ϕ maps R onto L . Multiplication in S is completely determined by that in R , and by k and ϕ , as follows ($x, y \in R$):

$$(1) \quad x\phi(y) = \phi(x)y = \phi(xy), \quad \phi(x)\phi(y) = kxy.$$

Moreover the order relation in L is determined by that in R , and by ϕ , as follows ($x, y \in R$):

$$(2) \quad \phi(x) < \phi(y) \text{ if and only if } \phi(x) \neq \phi(y) \text{ and } x > y.$$

We define the relation ρ in R by $x\rho y$ if and only if $\phi(x) = \phi(y)$. ρ is evidently a convex congruence relation in R , and L is order-anti-isomorphic with R/ρ . (There is no question here of product-isomorphism, since L is not closed under multiplication in S .) We note moreover that $\rho \leq \tau(k)$, i.e. $x\rho y$ implies $x\tau(k)y$. For $x\rho y$ means $\delta x = \delta y$, whence $kx = \delta^2 x = \delta^2 y = ky$.

Now let us travel the same road in the opposite direction, starting with an o.c.s.I, R , with endpoints $0 < u$, 0 the zero element and u the identity element of R . Let ρ be a convex congruence relation in R , and let k be an element of R such that $\rho \leq \tau(k)$. Let L be the set of congruence classes of $R \text{ mod } \rho$. Let ϕ be the canonical mapping of R onto L , so that $x\rho y$ if and only if $\phi(x) = \phi(y)$. Define an order relation in L by (2). (Thus L is an ordered set order-anti-isomorphic with R/ρ .)

We now identify 0 and $\phi(0)$, and let $S = L \cup R$, ordering S so as to preserve the order already defined in L and R , and so that $L < R \setminus \{0\}$. S has the least element $\delta = \phi(u)$. Define product in S by (1). To see that the definition is single-valued, let $\phi(x) = \phi(x')$ and $\phi(y) = \phi(y')$,

with x, y, x', y' in R . Then $x\phi(y) = \phi(xy)$ and $x\phi(y') = \phi(xy')$. But from ypy' we have $xypxy'$, i.e. $\phi(xy) = \phi(xy')$. Likewise, $\phi(x)\phi(y) = kxy$ and $\phi(x')\phi(y') = kx'y'$. But from xpx' and ypy' we have $xypx'y'$, and hence $kxy = kx'y'$ from $\rho \leq \tau(k)$.

Verification of associativity is routine; for example:

$$\begin{aligned}\phi(x)y \cdot \phi(z) &= \phi(xy) \cdot \phi(z) = k(xy) \cdot z, \\ \phi(x) \cdot y\phi(z) &= \phi(x) \cdot \phi(yz) = kx \cdot (yz); \\ \phi(x)\phi(y) \cdot \phi(z) &= kxy \cdot \phi(z) = \phi((kxy)x), \\ \phi(x) \cdot \phi(y)\phi(z) &= \phi(x) \cdot kyz = \phi(x(kyz)).\end{aligned}$$

We must show that every element of R is a conserver and every element of L and inverter. Here we shall verify the latter only; the former is similar.

Let $b \in L$, and let $s < t$ (s, t in S). We are to show $bs \geq bt$. Since ϕ maps R onto L , $b = \phi(a)$ with a in R . If $s \in R$ then $t \in R$ also, and $bs = \phi(a)s = \phi(as)$, $bt = \phi(a)t = \phi(at)$. From $s < t$ and $a, s, t \in R$ we have $as \leq at$, whence $\phi(as) \geq \phi(at)$ by (2), i.e. $bs \geq bt$. If $s \in L$ and $t \in R$, then $bs \in R$ and $bt \in L$, whence $bs \geq bt$. If $s \in L$ and $t \in L$, then $s = \phi(x)$, $t = \phi(y)$ for some x, y in R . From $s < t$ and (2) we have $x > y$. Now $bs = \phi(a)\phi(x) = kax$, $bt = \phi(a)\phi(y) = kay$. From $x > y$ and $k, a, x, y \in R$ we have $kax \geq kay$, i.e. $bs \geq bt$.

We summarize the foregoing in the following theorem.

THEOREM 2. *Let R be an ordered commutative semigroup (of the first kind) with endpoints $0 < u$, 0 being the zero element and u the identity element of R . Let ρ be a convex congruence relation in R , and k an element of R , such that xpy implies $kx = ky$. Let L be the set of congruence classes of $R \bmod \rho$, and let ϕ be the canonical mapping of R onto L . Order L by (2). Let $S = L \cup R$ with 0 and $\phi(0)$ identified, and order S so that $L < R \setminus \{0\}$. Define product in S by (1). Then S is an ordered commutative semigroup of the second kind with properties (a)–(d). Conversely, every ordered commutative semigroup of the second kind with properties (a)–(d) is obtainable from $R = [0, u]$ by the above construction.*

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