ORDERED COMMUTATIVE SEMIGROUPS OF THE SECOND KIND[1]

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By an *ordered commutative semigroup of the first kind* (abbreviated "o.c.s.I") we mean a system $S(\circ, <)$ consisting of a set S endowed with a binary operation $\circ$ and a binary relation $<$ such that the following axioms are satisfied.

I. *S is a commutative semigroup with respect to* $\circ$, i.e. the associative law, $a \circ (b \circ c) = (a \circ b) \circ c$, and the commutative law, $a \circ b = b \circ a$, hold (a, b, c in S).

II. *S is totally* ($=$*linearly*$=$*simply*) *ordered by* $<$.

III. *If a and b are elements of S such that $a<b$, then $a \circ c \leqq b \circ c$ for all c in S.*

Although this notion is very extensive, it does not include the ordered multiplicative system of all real numbers, in which multiplication by a negative number inverts the direction of any inequality. Let $S(\circ, <)$ satisfy just I and II above. An element c of S is called a *conserver* if $a<b$ (a, b in S) implies $c \circ a \leqq c \circ b$, and an *inverter* if $a<b$ implies $c \circ a \geqq c \circ b$. $S(\circ, <)$ will be called an *ordered commutative semigroup of the second kind* (abbreviated "o.c.s. II") if it satisfies I, II, and IV below.

IV. *Every element of S is either a conserver or an inverter, or both.*

Systems of this nature (in fact more general systems) have been investigated by Dov Tamari [**2**; **3**]. His definition of conserver (inverter) is more restrictive: $a<b$ implies $c \circ a < c \circ b$ ($c \circ a > c \circ b$). If c is a conserver or inverter in his sense, c is cancellable ($c \circ a = c \circ b$ implies $a = b$). I have taken the liberty of relaxing the definition so as to apply to noncancellable elements as well; the term "strict conserver (inverter)" may be used for his concept.

The main objective of the first part of the present paper is to show that the set P of conservers of S and the set Q of inverters of S are convex. (A subset A of S is *convex* if $a \in A$, $a' \in A$, and $a < x < a'$ imply $x \in A$.)

An ultimate objective is to construct all o.c.s.II's from o.c.s.I's. In the second part of the paper we give such a construction for a fairly restricted class of such semigroups. This was suggested by recent work of Haskell Cohen and L. I. Wade [**1**].

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NOTATION. We shall write ab instead of $a \circ b$. If A and B are subsets of S, then (1) AB denotes the set of all ab with a in A, b in B; (2) $A<B$ means $a<b$ for all a in A, b in B; (3) $A\backslash B$ means the set of elements of A not in B.

1. **Location of conservers and inverters.** Throughout this section, S will denote an o.c.s.II. P and Q will denote the sets of conservers and inverters, respectively, of S. The elements of $N=P\cap Q$ will be called *null elements*. Evidently $P^2\subseteq P$, $Q^2\subseteq P$, $PQ\subseteq Q$. In particular, P is a subsemigroup of S, and is an o.c.s.I. N is an ideal of S, i.e. $SN\subseteq N$, if it is not empty.

If S contains a zero element 0, i.e. $a0=0a=0$ for all a in S, we shall say that an element a of S is *positive* if $a>0$ and *negative* if $a<0$. We say that a and b *have the same sign* if they are both positive or both negative, and *have opposite sign* if one is positive and the other negative.

LEMMA 1. *Let S have a zero element* 0. *Let a and b be elements of S. If a and b are both conservers or both inverters, but have opposite sign, then $ab=0$. If a is a conserver and b an inverter, or vice-versa, and they have the same sign, then $ab=0$.*

PROOF. Suppose $a<0<b$. If a and b are both conservers, $a<0$ implies $ab\leqq 0b=0$, and $0<b$ implies $0=a0\leqq ab$, whence $ab=0$. If a and b are both inverters, $a<0$ implies $ab\geqq 0b=0$, and $0<b$ implies $0=a0\geqq ab$, and again we conclude $ab=0$.

Suppose a is a conserver and b an inverter, and $a<0$, $b<0$. $a<0$ implies $ab\geqq 0$ since b is an inverter. $b<0$ implies $ab\leqq 0$ since a is a conserver. Hence $ab=0$. The case $a>0$, $b>0$ is similar.

An element a of S is a null if and only if $ab=ac$ for all b, c in S. For if, say, $b<c$, then $ab\leqq ac$ because a is a conserver, and $ab\geqq ac$ because a is an inverter. From this we see that a is a null if and only if $ab=a^2$ for all b in S, i.e. $aS=\{a^2\}$.

LEMMA 2. *S contains a null element if and only if S contains a zero element* 0. *An element a of S is a null if and only if $aS=\{0\}$. The set N of null elements of S is a convex ideal containing* 0.

PROOF. Let a be a null element of S. Then a^2 is a zero element of S. For if b is any element of S, $a^2b=a(ab)=a^2$. Conversely, if S contains a zero element 0, then 0 is evidently a null element. Since the zero element is unique, we have shown that $a^2=0$ for any null a. Thus $aS=\{a^2\}=\{0\}$ for any null a, and conversely if $aS=\{0\}$ then a is clearly null.

We have noted above that N is an ideal. To show that it is convex, let $a \in N$, $b \in N$, $c \in S$, and $a < c < b$. Let $x \in S$. Then $0 = ax \leqq cx \leqq bx = 0$ if x is a conserver, and $0 = ax \geqq cx \geqq bx = 0$ if x is an inverter. In either event, $cx = 0$. Hence $cS = \{0\}$ and $c \in N$.

REMARK 1. The above proof of the convexity of N carries over to the following: *the set of annihilators of any element or subset of S is convex.*

LEMMA 3. *If N, $P \backslash N$, and $Q \backslash N$ are all nonempty, then $P \backslash N < N < Q \backslash N$ or $Q \backslash N < N < P \backslash N$.*

PROOF. Assume first that S contains arbitrarily large conservers. Then S cannot contain a positive non-null inverter. For suppose a is an inverter, and $a > 0$. We proceed to show that a is null.

Let $x > 0$. By hypothesis there exists a conserver $b \geqq x$. $ab = 0$ by Lemma 1, and hence $ax = 0$, from $0 < x \leqq b$ and Remark 1. Let $x < 0$. If x is an inverter, then $ax = 0$ by Lemma 1. If x is a conserver, let c be a conserver $\geqq a$; c exists, by assumption. Then $cx = 0$ by Lemma 1, and hence $ax = 0$ from $0 < a \leqq c$ and Remark 1.

Similarly we show: if S contains arbitrarily large inverters, then S cannot contain a positive non-null conserver; if S contains arbitrarily small conservers (inverters), then S cannot contain a negative non-null inverter (conserver). From these the lemma follows at once.

LEMMA 4. *If S does not contain a zero element, then $P < Q$ or $Q < P$.*

PROOF. Since $S = P \cup Q$ and $P \cap Q$ is empty, by Lemma 2, it suffices to show that P and Q are convex.

Suppose, by way of contradiction, that $a < b < c$ with a, c in P and b in Q. Since b is an inverter, $a < b < c$ implies $ab \geqq b^2 \geqq bc$. Since a is a conserver, $b < c$ implies $ab \leqq ac$. Since c is a conserver, $a < b$ implies $ac \leqq bc$. Hence $bc \leqq b^2 \leqq ab \leqq ac \leqq bc$, whence $bc = b^2$. But $bc \in QP \subseteq Q$, $b^2 \in Q^2 \subseteq P$, contrary to the fact that $P \cap Q$ is empty. The proof of the convexity of Q is similar.

We combine the foregoing lemmas into the following theorem.

THEOREM 1. *Let S be an ordered commutative semigroup of the second kind. Let P be the set of conservers, Q the set of inverters of S, and $N = P \cap Q$ the set of nulls of S. If N is empty, then $P < Q$ or $Q < P$. If N is not empty, then S has a zero element* 0, *$a \in N$ if and only if $aS = \{0\}$, and N is a convex ideal of S. If $P \backslash N$ and $Q \backslash N$ are not empty then N lies between them; in particular, the sets P, Q, N, $P \backslash N$ and $Q \backslash N$ are all convex.*

REMARK 2. If $Q \backslash N$ is empty then S is of the first kind. $P \backslash N$ need not be convex in this case. In fact we may take any two o.c.s.I's, S_1 and S_2, containing zero elements 0_1 and 0_2, respectively, with 0_1 the greatest element of S_1 and 0_2 the least element of S_2, and let S be their union, with 0_1 and 0_2 identified ($=0$, say). Order S so as to preserve the given order in S_1 and S_2 as already given, and make $S_1 < S_2 \backslash \{0\}$. Define products within S_1 and S_2 as already given, and for $a_1 \in S_1$, $a_2 \in S_2$, define $a_1a_2 = a_2a_1 = 0$. S becomes thereby an o.c.s.I. From Lemma 1 it follows that any o.c.s.I with interior zero element can be obtained in this fashion from its subsemigroups S_1 of all non-negative elements and S_2 of all non-positive elements.

REMARK 3. If $P\backslash N$ is empty, $Q\backslash N$ need not be convex. For example, let $S = \{a_1 < a_2 < c < 0 < d < b_1 < b_2\}$ with all products $=0$ except $a_ia_j = d$ and $b_ib_j = c$ (all $i, j = 1, 2$). We note that if $P\backslash N$ is empty, then $S^2 \subseteq N$, and so $S^3 = \{0\}$. For $S = Q$, whence $S^2 = Q^2 \subseteq P = P \cap Q = N$.

2. **Construction of a certain class of o.c.s.II's.** Let S and S' be o.c.s.II's. By a *homomorphism* of S into S' we mean a mapping f of S into S' satisfying (i) $f(ab) = f(a)f(b)$ for all a, b in S, and (ii) if $a < b$ then $f(a) \leqq f(b)$. By a *congruence relation* ρ in S we mean an equivalence relation such that $a\rho b$ implies $ac\rho bc$ for all c in S. ρ is called *convex* if each congruence class is convex. Let $[a]$ denote the congruence class to which a belongs (a in S). The *factor semigroup* S/ρ consists of all the classes $[a]$, with product defined by $[a]\cdot[b] = [ab]$. If ρ is convex, we can order S/ρ by defining $[a] < [b]$ if $[a] \neq [b]$ and $a < b$. S/ρ is then an o.c.s.II, and the *canonical mapping* $a \rightarrow [a]$ is a homomorphism of S onto S/ρ. If f is a homomorphism of S onto S', and we define $a\rho b$ if and only if $f(a) = f(b)$, then ρ is a convex congruence relation in S, and $[a] \rightarrow f(a)$ is an isomorphism of S/ρ onto S'.

An example of a convex congruence relation which we shall meet later is $\tau(k)$, where k is a fixed element of S, defined as follows: $a\tau(k)b$ if and only if $ka = kb$.

We proceed now to investigate the class of o.c.s.II's S satisfying the following conditions:

(a) *S is not of the first kind.*

(b) *S contains a zero element* 0, *not an endpoint.*

(c) *S has endpoints $\delta < u$, and u is the identity element of S* (i.e. *$ua = au = a$, all a in S*).

(d) *$L \subseteq \delta R$, where $L = [\delta, 0]$ and $R = [0, u]$.*

By (a), S contains a non-null inverter. S also contains the non-null conserver u. Since S contains an identity element, N must consist of 0 alone. By Theorem 1, 0 lies between the set $P\backslash N$ of non-null conservers and the set $Q\backslash N$ of non-null inverters. Since $u \in P\backslash N$, it follows that $R=P$ and $L=Q$. In particular, $\delta\in Q$, and so $\delta R\subseteq QP\subseteq Q=L$. (d) requires that δR be all of L.

The above notation is that of Cohen and Wade [1]. They are concerned with the case where S is a real interval, $x\circ y$ is a continuous function of the two variables x and y, and product $\circ$ is not necessarily commutative. (d) holds because the continuous mapping $x\rightarrow\delta x$ maps the connected set R onto the connected set δR containing 0 and δ, hence containing L. There are two main cases in [1], depending on whether $L^2\subseteq L$ or $L^2\subseteq R$. We are concerned only with the latter. Theorem 2 below may be regarded as an algebraic extract from Cohen and Wade's determination of S in this case. The present treatment will be self-contained.

R is a subsemigroup of S, and is an o.c.s.I, with the zero element at the lower end and the identity element at the upper end. We take the point of view that R is known, and our objective is to construct S in all possible ways from R.

First assume that S is given. Then $k=\delta^2\in R$. Let $\phi(x)=\delta x$ $(x\in R)$. By (d), ϕ maps R *onto* L. Multiplication in S is completely determined by that in R, and by k and ϕ, as follows $(x, y\in R)$:

$$(1)\qquad x\phi(y) = \phi(x)y = \phi(xy), \qquad \phi(x)\phi(y) = kxy.$$

Moreover the order relation in L is determined by that in R, and by ϕ, as follows $(x, y\in R)$:

$$(2)\qquad \phi(x) < \phi(y) \text{ if and only if } \phi(x) \neq \phi(y) \text{ and } x > y.$$

We define the relation ρ in R by $x\rho y$ if and only if $\phi(x)=\phi(y)$. ρ is evidently a convex congruence relation in R, and L is order-anti-isomorphic with R/ρ. (There is no question here of product-isomorphism, since L is not closed under multiplication in S.) We note moreover that $\rho\leqq\tau(k)$, i.e. $x\rho y$ implies $x\tau(k)y$. For $x\rho y$ means $\delta x=\delta y$, whence $kx=\delta^2x=\delta^2y=ky$.

Now let us travel the same road in the opposite direction, starting with an o.c.s.I, R, with endpoints $0<u$, 0 the zero element and u the identity element of R. Let ρ be a convex congruence relation in R, and let k be an element of R such that $\rho\leqq\tau(k)$. Let L be the set of congruence classes of R mod ρ. Let ϕ be the canonical mapping of R onto L, so that $x\rho y$ if and only if $\phi(x)=\phi(y)$. Define an order relation in L by (2). (Thus L is an ordered set order-anti-isomorphic with R/ρ.)

We now identify 0 and $\phi(0)$, and let $S=L\cup R$, ordering S so as to preserve the order already defined in L and R, and so that $L<R\backslash\{0\}$. S has the least element $\delta=\phi(u)$. Define product in S by (1). To see that the definition is single-valued, let $\phi(x)=\phi(x')$ and $\phi(y)=\phi(y')$,

with x, y, x', y' in R. Then $x\phi(y)=\phi(xy)$ and $x\phi(y')=\phi(xy')$. But from $y\rho y'$ we have $xy\rho xy'$, i.e. $\phi(xy)=\phi(xy')$. Likewise, $\phi(x)\phi(y)=kxy$ and $\phi(x')\phi(y')=kx'y'$. But from $x\rho x'$ and $y\rho y'$ we have $xy\rho x'y'$, and hence $kxy=kx'y'$ from $\rho\leqq\tau(k)$.

Verification of associativity is routine; for example:

$$\begin{aligned}\phi(x)y\cdot\phi(z) &= \phi(xy)\cdot\phi(z) = k(xy)\cdot z,\\ \phi(x)\cdot y\phi(z) &= \phi(x)\cdot\phi(yz) = kx\cdot(yz);\\ \phi(x)\phi(y)\cdot\phi(z) &= kxy\cdot\phi(z) = \phi((kxy)x),\\ \phi(x)\cdot\phi(y)\phi(z) &= \phi(x)\cdot kyz = \phi(x(kyz)).\end{aligned}$$

We must show that every element of R is a conserver and every element of L and inverter. Here we shall verify the latter only; the former is similar.

Let $b\in L$, and let $s<t$ (s, t in S). We are to show $bs\geqq bt$. Since ϕ maps R onto L, $b=\phi(a)$ with a in R. If $s\in R$ then $t\in R$ also, and $bs=\phi(a)s=\phi(as)$, $bt=\phi(a)t=\phi(at)$. From $s<t$ and a, s, $t\in R$ we have $as\leqq at$, whence $\phi(as)\geqq\phi(at)$ by (2), i.e. $bs\geqq bt$. If $s\in L$ and $t\in R$, then $bs\in R$ and $bt\in L$, whence $bs\geqq bt$. If $s\in L$ and $t\in L$, then $s=\phi(x)$, $t=\phi(y)$ for some x, y in R. From $s<t$ and (2) we have $x>y$. Now $bs=\phi(a)\phi(x)=kax$, $bt=\phi(a)\phi(y)=kay$. From $x>y$ and k, a, x, $y\in R$ we have $kax\geqq kay$, i.e. $bs\geqq bt$.

We summarize the foregoing in the following theorem.

THEOREM 2. *Let R be an ordered commutative semigroup* (*of the first kind*) *with endpoints $0<u$, 0 being the zero element and u the identity element of R. Let ρ be a convex congruence relation in R, and k an element of R, such that $x\rho y$ implies $kx=ky$. Let L be the set of congruence classes of R* mod ρ, *and let ϕ be the canonical mapping of R onto L. Order L by* (2). *Let $S=L\cup R$ with 0 and $\phi(0)$ identified, and order S so that $L<R\setminus\{0\}$. Define product in S by* (1). *Then S is an ordered commutative semigroup of the second kind with properties* (a)–(d). *Conversely, every ordered commutative semigroup of the second kind with properties* (a)–(d) *is obtainable from $R=[0, u]$ by the above construction.*

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