THE EQUIVALENCE OF TWO EXTENSIONS OF LEBESGUE AREA

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The area of a triangle in a Banach space was defined in [5] and then Lebesgue's definition for the area of a continuous function (surface) was applicable to a surface in a Banach space. If X is a continuous function on a closed simply-connected Jordan region J into a Banach space B let $L_B(X)$ denote its area in accordance with this definition. Let m be the space of bounded sequences [1]. It was shown that isometric functions[1] on J into m have the same L_m area. If X is continuous on J into a metric space M there is a continuous function x on J into m which is isometric with X. Define $L(X)$ to be equal to $L_m(x)$.

If X is continuous on J into a Euclidean space E, then $L_E(X)$ is the classical Lebesgue area of X. It was shown that $L(X)=L_E(X)$ and hence it seemed reasonable to call L an extension of Lebesgue area which applied to surfaces in a metric space.

Now let us consider those functions which are continuous on J into a Banach space B. There are two Lebesgue type areas available, L and L_B. Morrey's representation theorem [3] and the cyclic additivity theory were used to show that $L=L_B$ under a variety of conditions, in particular, as mentioned above, if $B=E$.

The hypothesis in Morrey's theorem that the functions range in E was too strong a restriction to enable us to conclude that $L(X) =L_B(X)$ for all Banach spaces B and continuous functions X on J into B. In [6] Morrey's theorem is extended to apply to surfaces in a metric space. We shall see that this version of Morrey's theorem can be used to show that L and L_B are equivalent, whenever L_B is applicable.

Let J be contained in the u, v plane. According to Cesari [2], a function X on J into E is a *D-mapping* if each component of X is A.C.T. in J^0, the interior of J, and if all of the partial derivatives are square summable over J^0. This definition can be generalized so as to apply to functions with range in a Banach space [6]. The property of being a D-mapping is invariant under an isometric transformation. If X is a D-mapping in B then X is of class $\overline{L}_B$ in the sense of [5] (see [4, IV. 4.33]).

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[1] If X and Y map J into metric spaces M and N respectively, then X and Y are *isometric* if $\text{dist}_M(X(u), X(v))=\text{dist}_N(Y(u), Y(v))$ for all $u, v\in J$.

The equivalence of L and L_B, whenever the latter is defined, is a result of the cyclic additivity theory, the fact that Fréchet equivalent functions have the same Lebesgue area, L or L_B, and Theorems 1 and 2 (from [5] and [6], respectively).

THEOREM 1. *If range $X \subset B$ then $L(X) \leqq L_B(X)$. If X is of class $\overline{L}_B$ then $L(X) = L_B(X)$.*

THEOREM 2. *If range $x \subset m$, if x is light, and if $L(x) < +\infty$, then there is a D-mapping y which is Fréchet equivalent to x.*

THEOREM 3. *If X is continuous on J into B then $L(X) = L_B(X)$.*

PROOF. We can use Theorem 1 and the cyclic additivity theory to assume, without loss of generality, that X is light and that $L(X) < +\infty$. Take x isometric with X, range $x \subset m$. According to Theorem 2 there is a D-mapping y which is Fréchet equivalent to x. For each $u \in J$ choose v to satisfy $x(v) = y(u)$ and define $Y(u) = X(v)$. (If v_1 and v_2 correspond to u then $x(v_1) = x(v_2)$ which implies that $X(v_1) = X(v_2)$.) It is easy to see that X and Y are Fréchet equivalent and that Y and y are isometric. Thus Y is a D-mapping and Theorem 1 enables us to conclude that $L_B(X) = L_B(Y) = L(Y) = L(X)$.

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