ON THE EXTREMA OF CERTAIN POLYNOMIALS

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Erdös has proposed the following four questions:

Let x be a real variable, and z a complex variable;

let $f(x) \equiv \prod_{k=1}^{n} (x - x_k)$, $-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$;

let $g(z) \equiv \prod_{k=1}^{n} (z - z_k)$, $z_k = e^{i\alpha_k}$, $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_n < 2\pi$;

let $y_s (x_s < y_s < x_{s+1},\ s = 1, 2, \cdots, n-1)$ be such that $f(y_s)$ is an extremum;

let $\beta_s (\alpha_s < \beta_s < \alpha_{s+1}$ for $s = 1, 2, \cdots, n-1$, and $\alpha_n < \beta_n < 2\pi)$ be such that $|g(e^{i\beta_s})|$ is a maximum.

For what choice of the x_k or z_k, respectively, is

$$\prod_{s=1}^{n-1} |f(y_s)| \text{ maximal?} \qquad \sum_{s=1}^{n-1} |f(y_s)| \text{ minimal?}$$

$$\prod_{s=1}^{n} |g(e^{i\beta_s})| \text{ maximal?} \qquad \sum_{s=1}^{n} |g(e^{i\beta_s})| \text{ minimal?}$$

The corresponding questions for the maximum values of $\sum |f(y_s)|$ and $\sum |g(e^{i\beta_s})|$ have been answered by P. Erdös[1] and R. Breusch;[2] the maximum is 2^n in both cases, with exceptions in the second case for small values of n, if the condition $|z_k| = 1$ is relaxed to $|z_k| \leqq 1$.

Erdös himself proved several years ago,[3] that $|f(1)| \cdot |f(-1)| \cdot \prod_{s=1}^{n-1} |f(y_s)|$ is maximal if $f(x)$ is a Legendre polynomial. In Part 2 of the present paper, it will be shown that

$$\frac{1}{2} |f(1)| + \frac{1}{2} |f(-1)| + \sum_{s=1}^{n-1} |f(y_s)| \geqq \frac{n}{2^{n-1}},$$

with equality if and only if $f(x)$ is a Tschebyschef polynomial. Parts 1 and 3 contain proofs that $\prod_{s=1}^{n} |g(e^{i\beta_s})| \leqq 2^n$, with equality if and only if $g(z) = z^n - e^{i\gamma}$, and that $\sum_{s=1}^{n} |g(e^{i\beta_s})| \geqq 2n$, with equality if and only if *either* $g(z) = z^n - e^{i\gamma}$, *or*, in the case of even n, $g(z) = (z^{n/2} - e^{i\gamma}) \cdot (z^{n/2} - e^{i\delta})$.

(1) Let $P(\alpha, \beta)$ stand for

$$\prod_{s=1}^{n} |g(e^{i\beta_s})| = \prod_{s=1}^{n} \prod_{k=1}^{n} |e^{i\beta_s} - e^{i\alpha_k}| = \prod_{k=1}^{n} |h(e^{i\alpha_k})|,$$

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[1] Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1169–1176.

[2] Bull. Amer. Math. Soc. vol. 53 (1947) pp. 982–986.

[3] Written Communication.

where $h(z)\equiv\prod_{s=1}^{n}(z-e^{i\beta_s})$. $|g(e^{i\beta_s})|$ is maximal; and if $P(\alpha,\beta)$ is to be maximal, then each $|h(e^{i\alpha_k})|$ must also be a maximum. Thus $\partial|g(e^{i\beta})|/\partial\beta=0$ for $\beta=\beta_s$, $s=1,2,\cdots,n$, and $\partial|h(e^{i\alpha})|/\partial\alpha=0$ for $\alpha=\alpha_k$, $k=1,2,\cdots,n$. Since $|e^{i\beta_s}-e^{i\alpha_k}|=2|\sin((\beta_s-\alpha_k)/2)|$, it follows that

$$\text{(I)}\qquad \sum_{k=1}^{n}\cot\frac{\alpha_k-\beta_s}{2}=0\qquad\text{for } s=1,2,\cdots,n$$

and

$$\text{(II)}\qquad \sum_{s=1}^{n}\cot\frac{\beta_s-\alpha_k}{2}=0\qquad\text{for } k=1,2,\cdots,n.$$

Now we consider the $2n$ differences

$$\Delta\alpha_k\equiv\alpha_{k+1}-\alpha_k \text{ for } k=1,2,\cdots,n-1,\quad \Delta\alpha_n\equiv 2\pi-\alpha_n,$$
$$\Delta\beta_s\equiv\beta_{s+1}-\beta_s \text{ for } s=1,2,\cdots,n-1,\quad \Delta\beta_n\equiv 2\pi+\beta_1-\beta_n.$$

Since the equations (I) and (II) are symmetric in the α_k and β_s, we may assume, without loss of generality, that $\Delta\alpha_1\equiv\alpha_2-\alpha_1=\alpha_2$ is the smallest of these $2n$ differences. By our choice of the coordinate directions, $\alpha_1=0$. It follows from (II), with $k=1$ and $k=2$, that

$$\sum_{s=1}^{n}\cot\frac{\beta_s}{2}=0,\quad\text{and}\quad\sum_{s=1}^{n}\cot\frac{\beta_s-\alpha_2}{2}=0.$$

We write the second equation in the form $\sum_{s=0}^{n-1}\cot((\beta_{s+1}-\alpha_2)/2)=0$, and subtract it from the first equation:

$$\cot\frac{\beta_n}{2}-\cot\frac{\beta_1-\alpha_2}{2}+\sum_{s=1}^{n-1}\left(\cot\frac{\beta_s}{2}-\cot\frac{\beta_{s+1}-\alpha_2}{2}\right)=0.$$

Since $\cot u-\cot v=\sin(v-u)/\sin u\cdot\sin v$, this may be written in the form

$$\text{(III)}\qquad \frac{\sin\dfrac{\beta_1-\alpha_2-\beta_n}{2}}{\sin\dfrac{\beta_n}{2}\cdot\sin\dfrac{\beta_1-\alpha_2}{2}}+\sum_{s=1}^{n-1}\frac{\sin\dfrac{\beta_{s+1}-\beta_s-\alpha_2}{2}}{\sin\dfrac{\beta_s}{2}\cdot\sin\dfrac{\beta_{s+1}-\alpha_2}{2}}=0.$$

$\sin\beta_s/2>0$ for $s=1,2,\cdots,n$; $\sin(\beta_{s+1}-\alpha_2)/2>0$ for $s=1,2,\cdots,n-1$; $\sin(\beta_{s+1}-\beta_s-\alpha_2)/2\geqq 0$ for $s=1,2,\cdots,n-1$, since by assumption

$$\beta_{s+1}-\beta_s\geqq\alpha_2.$$

Thus each term in the sum is non-negative. Also, $\sin(\beta_1-\alpha_2)/2<0$, and $\sin(\beta_1-\alpha_2-\beta_n)/2 \leqq 0$, since

$$\sin(\beta_1-\alpha_2-\beta_n)/2=-\sin(2\pi+\beta_1-\beta_n-\alpha_2)/2=-\sin(\Delta\beta_n-\alpha_2)/2,$$

and by assumption, $\Delta\beta_n \geqq \alpha_2$. Therefore the first term in (III) is also non-negative. It follows that each term must be zero, and this implies that

$$\Delta\beta_s = \alpha_2,$$

for $s=1, 2, \cdots, n$. Since $\sum_{s=1}^{n}\Delta\beta_s=2\pi$, this means that $n\cdot\alpha_2=2\pi$, $\alpha_2=2\pi/n$; and since $\Delta\alpha_k\geqq\alpha_2$, and $\sum_{k=1}^{n}\Delta\alpha_k=2\pi$, this implies that $\Delta\alpha_k=2\pi/n$, for $k=1, \cdots, n$. Thus the only possible maximum of $P(\alpha, \beta)$ occurs when $\alpha_k=(2\pi/n)\cdot(k-1)$, $g(z)=z^n-1$; or, if we drop the condition that $\alpha_1=0$, $g(z)$ must be of the form

$$g(z) = z^n - e^{i\gamma}.$$

In this case, $|g(e^{i\beta_s})|=2$, for $s=1, 2, \cdots, n$, and thus

$$P(\alpha, \beta) = 2^n.$$

(2) Let $f(x)$, x_k ($k=1, 2, \cdots, n$), and y_s ($s=1, 2, \cdots, n-1$) have the meaning as stated in the introduction. Then

$$f(y_s) = \int_{x_s}^{y_s} f'(x)dx = -\int_{y_s}^{x_{s+1}} f'(x)dx,$$

and therefore

$$2|f(y_s)| = \int_{x_s}^{x_{s+1}} |f'(x)|\,dx;$$

also

$$|f(-1)| = \int_{-1}^{x_1} |f'(x)|\,dx, \quad \text{and} \quad |f(1)| = \int_{x_n}^{1} |f'(x)|\,dx,$$

and thus

$$2S \equiv |f(-1)| + |f(1)| + 2\sum_{s=1}^{n-1} |f(y_s)| = \int_{-1}^{1} |f'(x)|\,dx.$$

In this development, use has been made of the fact that $f'(x)$ cannot change its sign except at $x=y_s$, $s=1, 2, \cdots, n-1$. Our problem is now to find a polynomial of degree n, with leading coefficient 1, and all roots real and between -1 and 1, such that $\int_{-1}^{1}|f'(x)|\,dx$ is as small as possible. With $x=\cos\theta$, $dx=-\sin\theta\cdot d\theta$, $2S=\int_0^{\pi}|f'(\cos\theta)|\cdot\sin\theta\cdot d\theta$

$=\int_0^\pi |f'(\cos\theta)\cdot\sin\theta|\cdot d\theta$. $f'(\cos\theta)=n\cos^{n-1}\theta+$lower powers of $\cos\theta$; $\cos^{n-1}\theta=2^{-(n-2)}\cdot\cos(n-1)\theta+$terms with $\cos k\theta$, $0\leqq k<n-1$. Thus

$$\begin{aligned} f'(\cos\theta)\cdot\sin\theta &= n\cdot 2^{-(n-2)}\cos(n-1)\theta\cdot\sin\theta+\cdots \\ &= n\cdot 2^{-(n-1)}\cdot[\sin n\theta-\sin(n-2)\theta] \\ &\quad + \text{terms with } \sin(m\theta),\ 0<m<n. \end{aligned}$$

Therefore $2S=\int_0^\pi |n\cdot 2^{-(n-1)}\sin n\theta+\cdots|\cdot d\theta$. Since

$$\frac{4}{\pi}\left|\sin(n\theta)+\frac{1}{3}\sin(3n\theta)+\frac{1}{5}\sin(5n\theta)+\cdots\right|=1$$

except for $\theta=k\pi/n$, $k=0,1,\cdots,n$, we may write $2S$ in the form

$$2S=\frac{4}{\pi}\int_0^\pi |n\cdot 2^{-(n-1)}\sin n\theta+\cdots|\cdot\left|\sin n\theta+\frac{1}{3}\sin 3n\theta+\cdots\right|\cdot d\theta$$

$$\geqq\frac{4}{\pi}\int_0^\pi\left(n\cdot 2^{-(n-1)}\sin n\theta+\cdots\right)\cdot\left(\sin n\theta+\frac{1}{3}\sin 3n\theta+\cdots\right)d\theta.$$

We expand the product, integrate term by term, and make use of the fact that if r and s are positive integers,

$$\int_0^\pi \sin(r\theta)\cdot\sin(s\theta)d\theta=\begin{cases}0 & \text{if } r\neq s,\\ \pi/2 & \text{if } r=s.\end{cases}$$

Therefore the only nonvanishing term in the integral is the one resulting from $\sin^2(n\theta)$, and thus

$$2S\geqq\frac{4}{\pi}n\cdot 2^{-(n-1)}\frac{\pi}{2},\qquad S\geqq\frac{n}{2^{n-1}}\cdot$$

Equality is possible if and only if the two factors under the integral change signs at exactly the same points. The second factor, and the first term in the first factor, change signs when $\theta=k\pi/n$, $k=1,2,\cdots,n-1$. Thus $h(\theta)$, the sum of the remaining terms of the first factor, must also vanish for $\theta=k\pi/n$, $k=1,2,\cdots,n-1$. Since all the terms of $h(\theta)$ are of the form $a_m\sin(m\theta)$, $h(\theta)$ is an odd function; it follows that $h(\theta)$ vanishes at the $2n$ points $k\pi/n$, $k=0,1,\cdots,2n-1$; but all the m are less than n, and therefore $h(\theta)$ can have not more than $2n-2$ zeros in $0\leqq\theta<2\pi$, unless $h(\theta)$ is identically equal to zero. Thus S can be equal to $n\cdot 2^{-(n-1)}$ only if

$$\begin{aligned} f'(\cos\theta)\cdot\sin\theta &= n\cdot 2^{-(n-1)}\cdot\sin(n\theta), \\ f(\cos\theta) &= 2^{-(n-1)}\cos(n\theta)+C, \\ f(x) &= 2^{-(n-1)}\cos(n\cdot\arccos x)+C=2^{-(n-1)}\cdot T_n(x)+C, \end{aligned}$$

where $T_n(x)$ is the nth Tschebyschef polynomial. If $|C| \leqq 2^{-(n-1)}$, then $f(x)$ is a polynomial of degree n, with leading coefficient 1, and which has n real zeros between -1 and 1; in this case $2^{-1}\int_{-1}^{+1}|f'(x)|\,dx$ really represents the sum

$$S = \frac{1}{2}\,|f(-1)| + \frac{1}{2}\,|f(1)| + \sum_{s=1}^{n-1}|f(y_s)|.$$

Thus it is seen that $S \geqq n/2^{n-1}$, with equality if and only if $f(x) = (1/2^{n-1})[T_n(x)+K]$, where K is a real constant of absolute value not greater than 1.

(3) Let $g(z)$, $z_k = e^{i\alpha_k}$, and β_s have the meanings assigned in the introduction. Let $g(z) = \sum_{r=0}^{n} a_r z^{n-r}$. Then $a_0 = 1$, and for $r > 0$,

$$a_r = (-1)^r S_r(z_k),$$

where $S_r(z_k)$ is the rth elementary symmetric function of the z_k.

$$S_{n-r}(z_k) = \prod_{k=1}^{n} z_k \cdot S_r(z_k^{-1}) = \prod_{k=1}^{n} z_k \cdot S_r(\bar{z}_k) = (-1)^r \prod_{k=1}^{n} z_k \cdot \bar{a}_r$$

where $\bar{a}_r$ is the conjugate of a_r. Thus $a_{n-r} = (-1)^{n-r} S_{n-r}(z_k) = (-1)^n \cdot \prod_{k=1}^{n} z_k \cdot \bar{a}_r$. If necessary, we rotate the coordinate system in such a way that

$$\sum_{k=1}^{n} \alpha_k \equiv \begin{cases} 0 \pmod{2\pi} \text{ if } n \text{ is even,} \\ \pi \pmod{2\pi} \text{ if } n \text{ is odd.} \end{cases}$$

Then $(-1)^n \cdot \prod z_k = 1$, and $a_{n-r} = \bar{a}_r$; in particular, if n is even, $a_{n/2}$ is real.

Thus $g(z) = z^{n/2} \cdot (z^{n/2} + z^{-n/2} + a_1 z^{n/2-1} + \bar{a}_1 z^{-(n/2-1)} + \cdots)$, and with $z = e^{i\theta}$, $a_r = |a_r| e^{i\sigma_r}$,

$$|g(z)| = 2\cdot\left|\cos\left(\frac{n}{2}\theta\right) + |a_1|\cdot\cos\left[\left(\frac{n}{2}-1\right)\theta + \sigma_1\right] + \cdots\right|$$
$$\equiv 2\cdot|h(\theta)|.$$

$h(\theta)$ is a real function of the real variable θ. Let $T \equiv \sum_{s=1}^{n} |g(e^{i\beta_s})| = 2\sum_{s=1}^{n} |h(\beta_s)|$. By a reasoning analogous to the one employed in Part 2, it is seen that

$$T = \int_0^{2\pi} |h'(\theta)|\,d\theta.$$

All the terms in $h'(\theta)$ are of the form $b_m \sin(m\theta/2 + \lambda_m)$, and all the m have the same parity as n. Thus

$$h'(2\pi+\theta) = \pm h'(\theta), \qquad \int_{2\pi}^{4\pi} |h'(\theta)|\,d\theta = \int_0^{2\pi} |h'(\theta)|\,d\theta$$

and

$$T = \frac{1}{2}\int_0^{4\pi} |h'(\theta)|\,d\theta = \frac{1}{2}\int_0^{4\pi} \left|\frac{n}{2}\sin\left(\frac{n}{2}\theta\right) + \cdots\right| d\theta,$$

or, with $\phi=\theta/2$,

$$T = \frac{1}{2}\int_0^{2\pi} |n\sin(n\phi) + \cdots|\,d\phi,$$

where the remaining terms are of the form $A_m \cos(m\phi)+B_m \sin(m\phi)$, with $0<m<n$. Again we multiply by

$$\frac{4}{\pi}\cdot\left|\sin(n\phi) + \frac{1}{3}\sin(3n\phi) + \frac{1}{5}\sin(5n\phi) + \cdots\right|.$$

$$\begin{aligned} T &= \frac{1}{2}\,\frac{4}{\pi}\int_0^{2\pi} |n\cdot\sin(n\phi) + \cdots|\cdot|\sin(n\phi) + \cdots|\cdot d\phi \\ &\geqq \frac{2}{\pi}\cdot\int_0^{2\pi} (n\sin(n\phi) + \cdots)(\sin(n\phi) + \cdots)\cdot d\phi \\ &= \frac{2}{\pi}\cdot n\cdot\pi = 2n. \end{aligned}$$

Equality is possible if and only if the two factors change signs at the same points, which is the case if and only if $h'(\theta)$ contains no term other than the first one. In this case, $h(\theta)=\cos(n\theta/2)$, or, if n is even, $h(\theta)=\cos(n\theta/2)+a_{n/2}$. Thus either $g(z)=z^n+1$, or, for even n, $g(z)=z^n+a_{n/2}\cdot z^{n/2}+1$. The last polynomial has all of its roots on the unit circle if and only if the real number $a_{n/2}$ is absolutely not greater than 2; in this case, $g(z)=(z^{n/2}+e^{i\gamma})\cdot(z^{n/2}+e^{-i\gamma})$. Dropping now the condition that $\sum\alpha_k\equiv 0$ or π, we see:

$$\sum_{s=1}^{n} |g(e^{i\beta_s})| \geqq 2n,$$

with equality if and only if either $g(z)=z^n-e^{i\gamma}$, or, when n is even, $g(z)=(z^{n/2}-e^{i\gamma})(z^{n/2}-e^{i\delta})$ (γ and δ real).

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