ON THE STRUCTURE OF MAXIMUM MODULUS ALGEBRAS

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Let U, K, and C denote the open unit disc, the closed unit disc, and the unit circumference, respectively. In [1], an algebra $\mathcal{R}$ of continuous complex functions defined on K was said to be a *maximum modulus algebra on K* if for every $f \in \mathcal{R}$

$$(1) \qquad \max_{z \in K} |f(z)| = \max_{z \in C} |f(z)|,$$

i.e., if every $f \in R$ attains its maximum modulus on C. As a matter of convenience, we shall, in this paper, abbreviate "maximum modulus algebra on K" to "M-algebra."

Examples of M-algebras which come to mind immediately are (a) the algebra $\mathcal{A}$ consisting of all functions which are continuous on K and analytic in U, (b) any subalgebra of $\mathcal{A}$, (c) any algebra $\mathcal{B}$ which is equivalent to an algebra of analytic functions via a homeomorphism of K, i.e., any algebra $\mathcal{B}$ with which there is associated a homeomorphism h of K into the complex plane, such that every $f \in \mathcal{B}$ is of the form

$$(2) \qquad f(z) = f^*(h(z)) \qquad (z \in K)$$

for some f^* which is continuous on $h(K)$ and analytic in $h(U)$.

Does this list contain all M-algebras? The results of [1] seemed to point toward a positive answer. In fact, the main theorem of [1], stated in somewhat different form, is as follows:

THEOREM 1. *Consider the following two conditions which an M-algebra $\mathcal{R}$ may satisfy:*

(i) *There is a function $h \in \mathcal{R}$ which is a homeomorphism of K.*

(ii) *$\mathcal{R}$ contains a nonconstant function $\phi \in \mathcal{A}$.*

Condition (i) *alone implies that $\mathcal{R}$ is equivalent to an algebra of analytic functions, via h. Conditions* (i) *and* (ii) *together imply that $\mathcal{R} \subset \mathcal{A}$.*

The next question that arises naturally is whether (ii) alone implies that $\mathcal{R} \subset \mathcal{A}$. That this is not so was shown by an example in [3]; there an M-algebra $\mathcal{R}'$ was constructed which was generated by two functions f and g, where f was analytic (and not constant) in U and g

Presented to the Society February 22, 1958; received by the editors March 5, 1958.

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was not analytic. It is clear, incidentally, that $\mathfrak{R}'$ cannot be equivalent to any algebra of analytic functions; for if $f(z)=f^*(h(z))$ and $g(z)=g^*(h(z))$, as in (2), with f^* and g^* analytic, then the analyticity of f implies that h is analytic (compare [**1**, p. 452]), and this forces g to be analytic.

The algebra $\mathfrak{R}'$ does not separate points on K (i.e., there exist $z_1\in K$, $z_2\in K$ such that $z_1\neq z_2$ but $\phi(z_1)=\phi(z_2)$ for every $\phi\in\mathfrak{R}'$). Thus the question arises whether the conclusion "$\mathfrak{R}\subset\mathfrak{A}$" of Theorem 1 can be rescued if we assume (ii) and some weakened form of (i), for instance, if we replace (i) by the requirement that $\mathfrak{R}$ should separate points on K (so that there is a canonical homeomorphism of K into the maximal ideal space of the Banach algebra $\overline{\mathfrak{R}}$, the uniform closure of $\mathfrak{R}$; we may assume without loss of generality that $\mathfrak{R}$ contains the constants [**1**, p. 450]). The answer, given in the present paper, settles the question raised in [**2**], and is again negative:

THEOREM 2. *There exists a finitely generated M-algebra $\mathfrak{R}$ such that*
(a) *$\mathfrak{R}$ separates points on K,*
(b) *$\mathfrak{R}$ contains nonconstant functions which are analytic in U, and*
(c) *$\mathfrak{R}$ contains functions which are not analytic in U.*

PROOF. Let P be a perfect, totally disconnected, bounded subset of the plane, whose two-dimensional Lebesgue measure is positive. Let Q be the set of all points (w_1, w_2, w_3, w_4) in the space of 4 complex variables (i.e., the 8-dimensional euclidean space E^8) such that $w_i\in P$ for $i=1, 2, 3, 4$; Q is the cartesian product $P\times P\times P\times P$, embedded in E^8 in a natural way. Note that both P and Q are homeomorphic to the Cantor set.

There exists a simple closed curve J in the plane such that $P\subset J$. Let D be the interior of J. The crux of the proof will be the construction of 4 complex continuous functions $h_1, \cdots, h_4$ on K, with the following properties:

(α) *There exists a subset H of C, homeomorphic to the Cantor set, such that the mapping*

$$z\to(h_1(z), h_2(z), h_3(z), h_4(z))$$

is one-to-one on H and maps H onto Q.

(β) *$h_1\in A$ and $h_1(K-H)\subset D$;*

(γ) *The set $\{h_2, h_3, h_4\}$ separates points on $K-H$;*

(δ) *There is an arc $L\subset U$ on which h_2 is constant.*

(We note that (δ) could be replaced by practically any condition which assures nonanalyticity.)

Once we have these functions, we can prove the theorem quite rapidly. Since P has positive measure, there exist nonconstant complex functions q_1, q_2, q_3 which are continuous in the plane, analytic in the complement of P (including the point at infinity), such that the set $\{q_1, q_2, q_3\}$ separates points in the plane; for the proof of this, see [**4**, pp. 826–827]. Let $\mathfrak{R}$ be the algebra generated by the functions f_{ij}, where

$$f_{ij}(z) = q_i(h_j(z)) \qquad (i = 1, 2, 3; j = 1, 2, 3, 4; z \in K). \tag{3}$$

Condition (β) implies that $f_{i1} \in A$; condition (δ) implies that $f_{i2} \notin A$; conditions (α), (β), (γ) together imply that the set $\{h_1, h_2, h_3, h_4\}$ separates points on K, and hence $\mathfrak{R}$ separates points on K. There only remains the verification that $\mathfrak{R}$ is an M-algebra.

Every member of $\mathfrak{R}$ is of the form

$$f(z) = g(f_{ij}(z)) = g(q_i(h_j(z))), \tag{4}$$

where g is a polynomial in 12 variables. Put

$$\phi(w_1, w_2, w_3, w_4) = g(q_i(w_j)). \tag{5}$$

If we keep w_2, w_3, w_4 fixed, then ϕ, as a function of w_1, is analytic in the complement of P. The maximum modulus theorem therefore implies that there is a point $w_1^* \in P$ such that

$$\left| \phi(w_1, w_2, w_3, w_4) \right| \leqq \left| \phi(w_1^*, w_2, w_3, w_4) \right| . \tag{6}$$

Keeping w_1^*, w_3, w_4 fixed, and then repeating this procedure twice more, we find that there is a point $(w_1^*, w_2^*, w_3^*, w_4^*) \in Q$ such that

$$\left| \phi(w_1, w_2, w_3, w_4) \right| \leqq \left| \phi(w_1^*, w_2^*, w_3^*, w_4^*) \right| \tag{7}$$

for all (w_1, w_2, w_3, w_4). By (α) there is a point $z^* \in H$ such that $h_j(z^*) = w_j^*$ $(j=1, \cdots, 4)$, and a glance at (4), (5), and (7) shows that

$$\left| f(z) \right| \leqq \left| f(z^*) \right| \tag{8}$$

for all $z \in K$.

Thus $\mathfrak{R}$ is an M-algebra, and Theorem 2 follows.

We now turn to the construction of the functions $h_1, \cdots, h_4$ and of the set H.

Let E be a perfect subset of C, of measure zero. There exist complex continuous functions $\phi_1, \cdots, \phi_4$, defined on E, such that the mapping

$$t \to (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)) \tag{9}$$

is a homeomorphism of E onto Q. By the theorem proved in [**3**], there exists a function $f_1 \in A$, such that $f_1(t) = \phi_1(t)$ for all $t \in E$ and such that $f(K)\subset D\cup J$. Let K' be the closed convex hull of E, let ψ be a conformal map of K onto K' (i.e., ψ is a homeomorphism of K onto K' which is conformal in the interior of K), and put $H=\psi^{-1}(E)$. Define

$$h_1(z) = f_1(\psi(z)) \qquad (z \in K), \tag{10}$$

and

$$h_j(z) = \phi_j(\psi(z)) \qquad (j = 2, 3, 4; z \in H). \tag{11}$$

Then condition (β) holds, and if we can extend h_2, h_3, h_4 from H to K so that (γ) and (δ) are satisfied, the proof will be complete, since (α) is implied by our choice of $\{\phi_j\}$.

Triangulate $K-H$; each compact subset of $K-H$ will be covered by a finite collection of triangles (some of these will be curvilinear), and every point of H will be a limit point of the set T of vertices. Pick two vertices t', $t''\in U$ which are joined by an edge of our triangulation, and define $h_j(t)$ for $j=2, 3, 4$ and $t\in T$ such that h_j is continuous on $H\cup T$, such that

$$h_2(t') = h_2(t'') = 0, \tag{12}$$

and such that the points $h(t)=(h_2(t), h_3(t), h_4(t))$ are in general position in E^6; i.e., no $m+2$ of these points lie in any linear m-space, for $m=1, \cdots, 4$.

Let Δ be one of our triangles, with vertices t_1, t_2, t_3. Define $h_j(z)$ for $z\in\Delta$ so that the mapping

$$z \to (h_2(z), h_3(z), h_4(z)) \tag{13}$$

is a homeomorphism of Δ onto the (rectilinear) triangle whose vertices are the points $h(t_1)$, $h(t_2)$, $h(t_3)$ in E^6.

The functions h_j are now extended to K and are continuous on K.

Since the points $h(t)$ are in general position, no two triangles whose vertices are among these points will intersect, except possibly in a common vertex or a common edge. It follows that condition (γ) is satisfied; and (12) shows that condition (δ) also holds, with the interval $[t', t'']$ for L.

This completes the proof of the theorem. It seems quite likely that another proof can be given by exhibiting an example with fewer generators; their number can perhaps be pushed down to 2, but different methods are needed for this.

In conclusion, we pose another problem:

Suppose $\mathfrak{R}$ *is an M-algebra such that* $\mathfrak{A}\cap\mathfrak{R}$ *separates points on K. Does it follow that* $\mathfrak{R}\subset\mathfrak{A}$?

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ON A CLASS OF UNIVERSAL ORDERED SETS

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An ordered set B is said to be $\aleph_\alpha$-universal if and only if every ordered set of power $\aleph_\alpha$ is similar to a subset of B. Let U_{ω_α} be the lexicographically ordered set of all sequences of 0's and 1's of type ω_α; and let H_α be the subset of U_{ω_α} consisting of all sequences $\{x_\xi\}_{\xi<\omega_\alpha}$ for which there is some $\xi_0<\omega_\alpha$ such that $x_{\xi_0}=1$ and, for $\xi>\xi_0$, $x_\xi=0$.

H_0, being countable, dense, and without first or last element, is similar to the set of rationals in their natural order, and therefore, is $\aleph_0$-universal. Sierpiński [2] has shown (as a direct consequence of his theorem that $H_{\alpha+1}$ is an $\eta_{\alpha+1}$-set) that, for any α, $H_{\alpha+1}$ is $\aleph_{\alpha+1}$-universal. Gillman [1] has given a demonstration that, for any limit ordinal α, H_α is $\aleph_\alpha$-universal. The purpose of this note is to give a very simple proof of these results, which does not depend on the ordinal α.

Theorem. *H_α is $\aleph_\alpha$-universal.*

Proof. Let A be an ordered set of power $\aleph_\alpha$. Fix some well-ordering $\{a_\beta\}_{\beta<\omega_\alpha}$ of A. Let $\prec$ denote the order in A. Define a function ϕ from A into H_α in the following way. Let a_τ be an element of A, and $\beta<\omega_\alpha$. Then the βth component $\phi_\beta(a_\tau)$ of $\phi(a_\tau)$ is defined by:

$$\phi_\beta(a_\tau) = \begin{cases} 1 & \text{if } \beta \leqq \tau \text{ and } a_\beta \preceq a_\tau, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let a_τ and a_σ be any elements of A, with $a_\tau \prec a_\sigma$. Clearly, if $\beta\leqq\sigma$, $\phi_\beta(a_\sigma)\geqq\phi_\beta(a_\tau)$. But, $\phi_\sigma(a_\sigma)=1$ and $\phi_\sigma(a_\tau)=0$. Hence, $\phi(a_\tau)$ pre-

Received by the editors March 27, 1958.