ON THE STRUCTURE OF MAXIMUM MODULUS ALGEBRAS

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Let $U$, $K$, and $C$ denote the open unit disc, the closed unit disc, and the unit circumference, respectively. In [1], an algebra $\mathcal{A}$ of continuous complex functions defined on $K$ was said to be a maximum modulus algebra on $K$ if for every $f \in \mathcal{A}$

$$\max_{z \in K} |f(z)| = \max_{z \in C} |f(z)|,$$

i.e., if every $f \in \mathcal{A}$ attains its maximum modulus on $C$. As a matter of convenience, we shall, in this paper, abbreviate “maximum modulus algebra on $K$” to “$M$-algebra.”

Examples of $M$-algebras which come to mind immediately are (a) the algebra $A$ consisting of all functions which are continuous on $K$ and analytic in $U$, (b) any subalgebra of $A$, (c) any algebra $\mathcal{B}$ which is equivalent to an algebra of analytic functions via a homeomorphism of $K$, i.e., any algebra $\mathcal{B}$ with which there is associated a homeomorphism $h$ of $K$ into the complex plane, such that every $f \in \mathcal{B}$ is of the form

$$f(z) = f^*(h(z)) \quad (z \in K)$$

for some $f^*$ which is continuous on $h(K)$ and analytic in $h(U)$.

Does this list contain all $M$-algebras? The results of [1] seemed to point toward a positive answer. In fact, the main theorem of [1], stated in somewhat different form, is as follows:

**Theorem 1.** Consider the following two conditions which an $M$-algebra $\mathcal{B}$ may satisfy:

(i) There is a function $h \in \mathcal{A}$ which is a homeomorphism of $K$.

(ii) $\mathcal{B}$ contains a nonconstant function $\phi \in A$.

Condition (i) alone implies that $\mathcal{A}$ is equivalent to an algebra of analytic functions, via $h$. Conditions (i) and (ii) together imply that $\mathcal{B} \subseteq \mathcal{A}$.

The next question that arises naturally is whether (ii) alone implies that $\mathcal{B} \subseteq \mathcal{A}$. That this is not so was shown by an example in [3]; there an $M$-algebra $\mathcal{B}'$ was constructed which was generated by two functions $f$ and $g$, where $f$ was analytic (and not constant) in $U$ and $g$
was not analytic. It is clear, incidentally, that \( \mathcal{R} \) cannot be equivalent to any algebra of analytic functions; for if \( f(z) = f^*(h(z)) \) and \( g(z) = g^*(h(z)) \), as in (2), with \( f^* \) and \( g^* \) analytic, then the analyticity of \( f \) implies that \( h \) is analytic (compare [1, p. 452]), and this forces \( g \) to be analytic.

The algebra \( \mathcal{R}' \) does not separate points on \( K \) (i.e., there exist \( z_1 \in K, z_2 \in K \) such that \( z_1 \neq z_2 \) but \( \phi(z_1) = \phi(z_2) \) for every \( \phi \in \mathcal{R}' \)). Thus the question arises whether the conclusion “\( \mathcal{R} \subset \mathcal{A} \)” of Theorem 1 can be rescued if we assume (ii) and some weakened form of (i), for instance, if we replace (i) by the requirement that \( \mathcal{R} \) should separate points on \( K \) (so that there is a canonical homeomorphism of \( K \) into the maximal ideal space of the Banach algebra \( \mathcal{R} \), the uniform closure of \( \mathcal{R} \); we may assume without loss of generality that \( \mathcal{R} \) contains the constants [1, p. 450]). The answer, given in the present paper, settles the question raised in [2], and is again negative:

**Theorem 2.** There exists a finitely generated \( M \)-algebra \( \mathcal{R} \) such that

\begin{enumerate}
  \item \( \mathcal{R} \) separates points on \( K \),
  \item \( \mathcal{R} \) contains nonconstant functions which are analytic in \( U \), and
  \item \( \mathcal{R} \) contains functions which are not analytic in \( U \).
\end{enumerate}

**Proof.** Let \( P \) be a perfect, totally disconnected, bounded subset of the plane, whose two-dimensional Lebesgue measure is positive. Let \( Q \) be the set of all points \((w_1, w_2, w_3, w_4)\) in the space of 4 complex variables (i.e., the 8-dimensional euclidean space \( E^8 \)) such that \( w_i \in P \) for \( i = 1, 2, 3, 4 \); \( Q \) is the cartesian product \( P \times P \times P \times P \), embedded in \( E^8 \) in a natural way. Note that both \( P \) and \( Q \) are homeomorphic to the Cantor set.

There exists a simple closed curve \( J \) in the plane such that \( P \subset J \). Let \( D \) be the interior of \( J \). The crux of the proof will be the construction of 4 complex continuous functions \( h_1, \ldots, h_4 \) on \( K \), with the following properties:

\begin{enumerate}
  \item There exists a subset \( H \) of \( C \), homeomorphic to the Cantor set, such that the mapping
    \[ z \rightarrow (h_1(z), h_2(z), h_3(z), h_4(z)) \]
  is one-to-one on \( H \) and maps \( H \) onto \( Q \).
  \item \( h_1 \in A \) and \( h_1(K - H) \subset D \); \( h_1 \in A \) and \( h_1(K - H) \subset D \);
  \item The set \( \{h_2, h_3, h_4\} \) separates points on \( K - H \);
  \item There is an arc \( L \subset U \) on which \( h_2 \) is constant.
\end{enumerate}

(We note that (\( \delta \)) could be replaced by practically any condition which assures nonanalyticity.)

Once we have these functions, we can prove the theorem quite
rapidly. Since $P$ has positive measure, there exist nonconstant complex functions $q_1, q_2, q_3$ which are continuous in the plane, analytic in the complement of $P$ (including the point at infinity), such that the set $\{q_1, q_2, q_3\}$ separates points in the plane; for the proof of this, see [4, pp. 826–827]. Let $\mathfrak{A}$ be the algebra generated by the functions $f_{ij}$, where

$$f_{ij}(z) = q_i(h_j(z)) \quad (i = 1, 2, 3; j = 1, 2, 3, 4; z \in K).$$

Condition $(\beta)$ implies that $f_{ii} \in \mathcal{A}$; condition $(\delta)$ implies that $f_{iz} \in \mathcal{A}$; conditions $(\alpha), (\beta), (\gamma)$ together imply that the set $\{h_1, h_2, h_3, h_4\}$ separates points on $K$, and hence $\mathfrak{A}$ separates points on $K$. There only remains the verification that $\mathfrak{A}$ is an $M$-algebra.

Every member of $\mathfrak{A}$ is of the form

$$f(z) = g(f_{ij}(z)) = g(q_i(h_j(z))),$$

where $g$ is a polynomial in 12 variables. Put

$$\phi(w_1, w_2, w_3, w_4) = g(q_i(w_j)).$$

If we keep $w_2, w_3, w_4$ fixed, then $\phi$, as a function of $w_1$, is analytic in the complement of $P$. The maximum modulus theorem therefore implies that there is a point $w_1^* \in P$ such that

$$|\phi(w_1, w_2, w_3, w_4)| \leq |\phi(w_1^*, w_2, w_3, w_4)|.$$ 

Keeping $w_1^*, w_3, w_4$ fixed, and then repeating this procedure twice more, we find that there is a point $(w_1^*, w_2^*, w_3^*, w_4^*) \in Q$ such that

$$|\phi(w_1, w_2, w_3, w_4)| \leq |\phi(w_1^*, w_2^*, w_3^*, w_4^*)|$$

for all $(w_1, w_2, w_3, w_4)$. By $(\alpha)$ there is a point $z^* \in H$ such that $h_j(z^*) = w_j^* \quad (j = 1, \cdots, 4)$, and a glance at (4), (5), and (7) shows that

$$|f(z)| \leq |f(z^*)|$$

for all $z \in K$.

Thus $\mathfrak{A}$ is an $M$-algebra, and Theorem 2 follows.

We now turn to the construction of the functions $h_1, \cdots, h_4$ and of the set $\mathcal{H}$.

Let $E$ be a perfect subset of $C$, of measure zero. There exist complex continuous functions $\phi_1, \cdots, \phi_4$, defined on $E$, such that the mapping

$$t \mapsto (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t))$$

is a homeomorphism of $E$ onto $Q$. By the theorem proved in [3], there exists a function $f_1 \in \mathcal{A}$, such that $f_1(t) = \phi_1(t)$ for all $t \in E$ and
such that $f(K) \subset D \cup J$. Let $K'$ be the closed convex hull of $E$, let $\psi$ be a conformal map of $K$ onto $K'$ (i.e., $\psi$ is a homeomorphism of $K$ onto $K'$ which is conformal in the interior of $K$), and put $H = \psi^{-1}(E)$. Define

$$h_1(z) = f_1(\psi(z)) \quad (z \in K),$$

and

$$h_j(z) = \phi_j(\psi(z)) \quad (j = 2, 3, 4; z \in H).$$

Then condition (β) holds, and if we can extend $h_2, h_3, h_4$ from $H$ to $K$ so that (γ) and (δ) are satisfied, the proof will be complete, since (α) is implied by our choice of $\{\phi_j\}$.

Triangulate $K - H$; each compact subset of $K - H$ will be covered by a finite collection of triangles (some of these will be curvilinear), and every point of $H$ will be a limit point of the set $T$ of vertices. Pick two vertices $t', t'' \in U$ which are joined by an edge of our triangulation, and define $h_j(t)$ for $j = 2, 3, 4$ and $t \in T$ such that $h_j$ is continuous on $H \cup T$, such that

$$h_2(t') = h_2(t'') = 0,$$

and such that the points $h(t) = (h_2(t), h_3(t), h_4(t))$ are in general position in $E^6$; i.e., no $m + 2$ of these points lie in any linear $m$-space, for $m = 1, \ldots, 4$.

Let $\Delta$ be one of our triangles, with vertices $t_1, t_2, t_3$. Define $h_j(z)$ for $z \in \Delta$ so that the mapping

$$z \to (h_2(z), h_3(z), h_4(z))$$

is a homeomorphism of $\Delta$ onto the (rectilinear) triangle whose vertices are the points $h(t_1), h(t_2), h(t_3)$ in $E^6$.

The functions $h_j$ are now extended to $K$ and are continuous on $K$.

Since the points $h(t)$ are in general position, no two triangles whose vertices are among these points will intersect, except possibly in a common vertex or a common edge. It follows that condition (γ) is satisfied; and (12) shows that condition (δ) also holds, with the interval $[t', t'']$ for $L$.

This completes the proof of the theorem. It seems quite likely that another proof can be given by exhibiting an example with fewer generators; their number can perhaps be pushed down to 2, but different methods are needed for this.

In conclusion, we pose another problem:

*Suppose $\mathcal{A}$ is an $M$-algebra such that $\mathcal{A} \cap \mathcal{R}$ separates points on $K$. Does it follow that $\mathcal{A} \subset \mathcal{A}$?*
ON A CLASS OF UNIVERSAL ORDERED SETS

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An ordered set $B$ is said to be $\aleph_\alpha$-universal if and only if every ordered set of power $\aleph_\alpha$ is similar to a subset of $B$. Let $U_\omega$ be the lexicographically ordered set of all sequences of 0's and 1's of type $\omega$; and let $H_\alpha$ be the subset of $U_\omega$ consisting of all sequences $\{x_\xi\}_{\xi<\omega}$ for which there is some $\xi_0<\omega$ such that $x_{\xi_0}=1$ and, for $\xi>\xi_0$, $x_\xi=0$.

$H_0$, being countable, dense, and without first or last element, is similar to the set of rationals in their natural order, and therefore, is $\aleph_0$-universal. Sierpiński [2] has shown (as a direct consequence of his theorem that $H_{\alpha+1}$ is an $\eta_{\alpha+1}$-set) that, for any $\alpha$, $H_{\alpha+1}$ is $\aleph_{\alpha+1}$-universal. Gillman [1] has given a demonstration that, for any limit ordinal $\alpha$, $H_\alpha$ is $\aleph_\alpha$-universal. The purpose of this note is to give a very simple proof of these results, which does not depend on the ordinal $\alpha$.

**Theorem.** $H_\alpha$ is $\aleph_\alpha$-universal.

**Proof.** Let $A$ be an ordered set of power $\aleph_\alpha$. Fix some well-ordering $\{a_\beta\}_{\beta<\omega}$ of $A$. Let $<$ denote the order in $A$. Define a function $\phi$ from $A$ into $H_\alpha$ in the following way. Let $a_\tau$ be an element of $A$, and $\beta<\omega$. Then the $\beta$th component $\phi_\beta(a_\tau)$ of $\phi(a_\tau)$ is defined by:

$$
\phi_\beta(a_\tau) = \begin{cases} 
1 & \text{if } \beta \leq \tau \text{ and } a_\beta \leq a_\tau, \\
0 & \text{otherwise.}
\end{cases}
$$

Now, let $a_\tau$ and $a_\sigma$ be any elements of $A$, with $a_\tau< a_\sigma$. Clearly, if $\beta \leq \sigma$, $\phi_\beta(a_\sigma) \geq \phi_\beta(a_\tau)$. But, $\phi_\beta(a_\sigma)=1$ and $\phi_\beta(a_\tau)=0$. Hence, $\phi(a_\tau)$ pre-

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