ON A CLASS OF FIXED-POINT-FREE GRAPHS[1]

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A number of papers [**1**; **2**; **3**; **4**] have dealt with the construction of finite graphs X whose automorphism group $G(X)$ is isomorphic to a given finite group G. Examination of the graphs constructed in these papers shows the following two facts. (1) The graphs X have the property that for any two vertices x and y of X there is at most one $\phi \in G(X)$ which sends x into y ((1) is precisely the fact which is used in [**1**] and [**2**] to prove that $G(X)$ actually is isomorphic to G). (2) The graphs X are modifications of Cayley color-groups of the given group G with respect to some set H of generators of G (cf. Definition 2 below) in the sense that the vertices and edges of the color-group are replaced by certain graphs (cf. Definition 3). It is the purpose of the present note to show that (1) implies (2) (Theorem 3).

By a *graph* X we mean a *finite* set V together with a set E of unordered pairs of distinct elements of V. We shall indicate unordered pairs by brackets. The elements of V are called the *vertices* of X, the elements of E the *edges* of X. To distinguish between different graphs we shall always write $V(X)$ for V, and $E(X)$ for E. By $G(X)$ we denote the *automorphism group* of X. We shall consider the elements of $G(X)$ as permutations of the vertices of X.

DEFINITION 1. A graph X is *strongly fixed-point-free*, if $G(X) \neq \{1\}$, and $\phi x \neq x$ for every $x \in V(X)$ and every $\phi \in G(X) - \{1\}$, where 1 is the identity of $G(X)$.

If X is strongly fixed-point-free, then clearly X is fixed-point-free (cf. [**4**, §1]). For a graph X with $G(X) \neq \{1\}$ the following are equivalent: (i) X is strongly fixed-point-free; (ii) if $\phi \in G(X)$ and $\phi x = x$ for some $x \in V(X)$, then $\phi = 1$; (iii) given $x \in V(X)$ and $y \in V(X)$ there is at most one $\phi \in G(X)$ such that $\phi x = y$.

LEMMA 1. *Let X be a strongly fixed-point-free graph. Then X is either connected or it consists of exactly two components X_1, X_2 such that $X_1 \cong X_2$, and $G(X_1) \cong G(X_2) = \{1\}$.*

PROOF. Suppose that X is disconnected. Let X_i, $i = 1, \cdots, n$, $n \geqq 2$, be the components of X. Then at least two of the X_i are isomorphic. Otherwise $\phi | X_i \in G(X_i)$ for every $\phi \in G(X)$ and $i = 1, \cdots, n$.

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Since X is strongly fixed-point-free, $G(X)\neq\{1\}$, hence at least one $G(X_{i_0})\neq\{1\}$, $1\leqq i_0\leqq n$. Let $\phi_0\in G(X_{i_0})$, $\phi_0\neq 1$. Define ϕ: $V(X)\to V(X)$ by $\phi x=\phi_0 x$ or if $x\in V(X_{i_0})$, $\phi x=x$ otherwise. Clearly $\phi\in G(X)$, but this is a contradiction against the fact that X is strongly fixed-point-free. Hence $X_1\cong X_2$, say.

Now assume that X has more than two components. Let ψ be an isomorphism of X_1 onto X_2. Then ϕ': $V(X)\to V(X)$ given by $\phi' x=\psi x$ if $x\in V(X_1)$, $\phi' x=\psi^{-1}x$ if $x\in V(X_2)$, $\phi' x=x$ otherwise, is an automorphism of X. As above this is a contradiction against X being strongly fixed-point-free.

Finally, suppose that $G(X_1)\cong G(X_2)\neq\{1\}$. Let $\phi_1\in G(X_1)-\{1\}$. Then ϕ'': $V(X)\to V(X)$ given by $\phi'' x=\phi_1 x$ if $x\in V(X_1)$, $\phi'' x=x$ if $x\in V(X_2)$ is an automorphism of X, a contradiction.

LEMMA 2. *Let X be a strongly fixed-point-free graph, and let $x\in V(X)$. Then there exist at most two edges of X which are similar and incident with x.*

PROOF. Let $e_i=[x, x_i]\in E(X)$, $i=0, 1, 2$, be similar. Then there exist $\phi_i\in G(X)$ such that $e_i=\phi_i e_0$, $i=1, 2$. Since X is strongly fixed-point-free this implies that (1) $\phi_i x=x_i$, and (2) $\phi_i x_0=x$, $i=1, 2$. (2) implies $\phi_1=\phi_2$, hence from (1), $x_1=x_2$, i.e. $e_1=e_2$.

LEMMA 3. *Let X be a connected fixed-point-free graph such that $E(X)$ consists of more than one edge, and $G(X)$ acts transitively on $E(X)$. Then X is cyclically connected.*

PROOF. Assume the contrary. Let B be a block[2] of X which contains exactly one cut vertex x of X, and let $e\in E(B)$ be incident with x. *Case* (1): $E(B)$ consists of the single edge e. In this case X is a star[3] by the transitivity of $G(X)$, and hence is not fixed-point-free. *Case* (2): $E(B)$ consists of more than one edge. Then there is an edge of B which is not incident with x, and hence with no cut vertex of X. But this contradicts the hypothesis that all edges of X are similar.

THEOREM 1. *Let X be a strongly fixed-point-free graph such that* (i) $E(X)\neq\square$, *and* (ii) $G(X)$ *acts transitively on* $E(X)$. *Then X is isomorphic to the graph Z_0 given by* $V(Z_0)=\{x_1, x_2\}$, $E(Z_0)=\{[x_1, x_2]\}$.

PROOF. We shall show that if X is strongly fixed-point-free, and contains more than one edge, then $G(X)$ does not act transitively on

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[2] A *block* of a connected graph X is a maximal connected subgraph B of X such that no vertex of B is a cut vertex of B.

[3] A *star* is a graph X such that $V(X)=\{x_0, \cdots, x_n\}$, $E(X)=\{[x_0, x_i], i=1, \cdots, n\}$.

$E(X)$. Assume the contrary. Then X must be connected. Otherwise it follows from Lemma 1 that X consists of two components X_1, X_2 such that $X_1 \cong X_2$ and $G(X_1) \cong G(X_2) = \{1\}$. But $G(X_1) \cong G(X_2) = \{1\}$ is impossible since by assumption X contains more than one edge, and all edges of X are similar. From Lemma 3 we then infer that X is cyclically connected. Hence the degree of every vertex of X is $\geqq 2$. Since all edges of X are similar, all edges incident with a given vertex $x \in V(X)$ are similar. Hence by Lemma 2 the degree of x in X is $\leqq 2$. It follows that X is a circuit, but a circuit is not strongly fixed-point-free.

DEFINITION 2. Let G be a finite group, and let H be a subset of G which does not contain the identity of G. By the *color-group* $X_{G,H}$ *of* G *with respect* to H we mean the graph given by $V(X_{G,H}) = G$, $E(X_{G,H}) = \{[g, gh] \mid g \in G, h \in H\}$.

It is known that $X_{G,H}$ is connected if and only if H is a set of generators of G.

LEMMA 4. *Given a graph X a n.a.s.c. for the existence of a group G and a subset H of G such that $X \cong X_{G,H}$ is that $G(X)$ contain a subgroup G_0 of order $\alpha_0(X)$ which acts transitively on $V(X)$. In that case $G = G_0$.*

PROOF. *Necessity.* Suppose $X = X_{G,H}$. Define $\eta: G \rightarrow G(X_{G,H})$ by $(\eta g)g' = gg'$, (where $g, g' \in G$), then clearly η is a monomorphism. Hence Im η is of order $[G: 1] = \alpha_0(X_{G,H}) = \alpha_0(X)$. Also, Im η acts transitively on $V(X) = V(X_{G,H}) = G$.

Sufficiency. Suppose $G(X)$ contains a subgroup G_0 of order $\alpha_0(X)$ which acts transitively on $V(X)$. Let $x_0, x \in V(X)$. Then there is exactly one $\phi_x \in G_0$ such that $\phi_x x_0 = x$. Let $x_1, \cdots, x_n$ be those vertices of X which are joined with x_0. Let $H = \{\phi_{x_1}, \cdots, \phi_{x_n}\}$, and form $X_{G_0,H}$. Define $\epsilon: V(X_{G_0,H}) \rightarrow V(X)$ by $\epsilon\phi_x = x$, $\phi_x \in V(X_{G_0,H}) = G_0$. We have to show that ϵ is an isomorphism of $X_{G_0,H}$ onto X. Let $[\phi_x, \phi_x\phi_{x_i}] \in E(X_{G_0,H})$, $\phi_x \in G_0$, $\phi_{x_i} \in H$, then $\epsilon[\phi_x, \phi_x, \phi_{x_i}] = [\phi_x x_0, \phi_x x_i] \in E(X)$ since $[x_0, x_i] \in E(X)$. Hence ϵ preserves incidence. Conversely, let $[x, y] \in E(X)$; then $\phi_x^{-1}[x, y] = [x_0, \phi_x^{-1} y] \in E(X)$. Hence $\phi_x^{-1} y = x_i = \phi_{x_i} x_0$, $1 \leqq i \leqq n$. Hence $[x, y] = \epsilon[\phi_x, \phi_x\phi_{x_i}]$, and $[\phi_x, \phi_x\phi_{x_i}] \in E(X_{G_0,H})$. Thus ϵ maps $E(X_{G_0,H})$ onto $E(X)$. That ϵ is one-one follows from the fact that $[G_0: 1] = \alpha_0(X)$. Hence ϵ is an isomorphism onto.

THEOREM 2. *Let X be a strongly fixed-point-free graph such that $G(X)$ acts transitively on $V(X)$. Then X is isomorphic to the graph Y_0 with $V(Y_0) = \{x_1, x_2\}$, $E(Y_0) = \square$, or else X is connected and isomorphic to a color-group of $G(X)$ with respect to some set H of generators of $G(X)$.*

PROOF. Suppose X is disconnected. Then by Lemma 1, X consists of exactly two components X_1, X_2 such that $X_1 \cong X_2$ and $G(X_1) \cong G(X_2) = \{1\}$. The last condition is compatible with the requirement that $G(X)$ act transitively on $V(X)$ if and only if X_1 and X_2 each consist of a single vertex, i.e. if $X \cong Y_0$.

If X is connected, Theorem 2 is an immediate consequence of Lemma 4.

DEFINITION 3. Suppose given a finite group G and a subset H of G such that $1 \notin H$. Let Y be a connected graph such that for each $h \in H$ there exist two subsets $P_h = \{p_1^{(h)}, \cdots, p_{r_h}^{(h)}\}$, $Q_h = \{q_1^{(h)}, \cdots, q_{r_h}^{h}\}$ of $V(Y)$ with the property that $P_h \neq P_{h'}$ or $Q_h \neq Q_{h'}$ whenever $h \neq h'$. Suppose furthermore that $G'(Y) = \{1\}$, where $G'(Y)$ is that subgroup of $G(Y)$ which leaves the elements of P_h and Q_h, $h \in H$, individually invariant. For each $g \in G$ take a copy Y_g of Y, and let η_g be an isomorphism of Y onto Y_g. Form a graph X by identifying the vertices $\eta_g p_j^{(h)}$ and $\eta_{gh} q_j^{(h)}$, $j = 1, \cdots, r_h$, $g \in G$, $h \in H$. Then X is called the *generalized color-group of G with respect to H and Y.*

THEOREM 3. *Let X be a connected strongly fixed-point-free graph. Then X is isomorphic to a generalized color-group of $G(X)$ with respect to some set H of generators of $G(X)$.*

PROOF. Let Y be a subgraph of X such that (i) Y is connected; (ii) no two distinct edges of Y are similar in X; (iii) Y is maximal with respect to (i), (ii). Then

$$E(X) = \bigcup_{\phi \in G(X)} E(\phi Y). \tag{A}$$

We prove this by showing that $E(Y)$ meets every similarity class of $E(X)$. Assume the contrary. Then among those edges of X which are not similar to any edge of Y select one, e_0, whose distance[4] from $E(Y)$ is minimal. This is possible because X is connected. Let e be an edge of Y which is closest to e_0, and let P be a path of minimal length of X containing both e and e_0. Let e'_0 be that edge of P which is adjacent to e_0. Since e_0' is nearer to E than e_0 it follows that e_0' is similar to some edge $e' \in E(Y)$. Hence there is a $\phi \in G(X)$ such that $\phi e_0' = e'$. $\phi e_0 \notin E(Y)$ is adjacent to e'; hence the minimal subgraph Y_0 of X with $E(Y_0) = E(Y) \cup \{\phi e_0\}$ has properties (i) and (ii), i.e. Y is not maximal, a contradiction.

[4] By the *distance* of two edges e, $e' \in E(X)$ is meant $d(e, e') = \min \alpha_1(P) - 1$ the minimum taken over all paths P of X such that $e, e' \in E(P)$. The distance of an edge $e \in E(X)$ from a subset $E \subset E(X)$ is defined as $d(e, E) = \min d(e, e')$, the minimum taken over all $e' \in E$.

Let X be such that $G(X)$ acts transitively on $E(X)$. Then by Theorem 1, $X \cong Z_0$, and clearly Z_0 is a color-group of $G(Z_0) = \{1, (x_1, x_2)\}$.

Let X be such that $G(X)$ does not act transitively on $E(X)$, and let $\phi, \psi \in G(X)$, $\phi \neq \psi$. Then $\phi Y \neq \psi Y$. It suffices to show that if $\psi \neq 1$, then $\psi Y \neq Y$. Assume the contrary. Then for each $e \in E(Y)$ there is an $e' \in E(Y)$ such that $e = \psi e'$. If $e \neq e'$ this is a contradiction against (ii). Hence $e = \psi e$ for all $e \in E(Y)$. Let $e = [x, y] \in E(Y)$. Then $e = \psi e$ implies (1) $\psi x = x$, $\psi y = y$, or (2) $\psi x = y$, $\psi y = x$. (1) means $\psi = 1$, a contradiction. Hence (2) must hold. Now suppose that $E(Y)$ contains at least two edges. Since Y is connected (property (i)), there is a $z \in V(Y)$, $z \neq y$, such that $[x, z] \in E(Y)$. Then by the same argument as above $\psi x = z$, $\psi z = x$. Therefore by (2), $z = y$, a contradiction. Hence $E(Y) = \{e\}$. But this means that $G(X)$ acts transitively on $E(X)$, again a contradiction.

Now define a graph X^* as follows: $V(X^*) = \{\phi Y \mid \phi \in G(X)\}$, $E(X^*) = \{[\phi Y, \phi' Y] \mid \phi Y \neq \phi' Y,\ V(\phi Y) \cap V(\phi' Y) \neq \square\}$. It follows from the preceding paragraph that $\alpha_0(X^*) = [G(X) : 1]$. By (A), X^* is connected.

Define a function j: $G(X) \to G(X^*)$ by $(j\phi)(\psi Y) = \phi\psi Y$, $\phi, \psi \in G(X)$. Then clearly j is a monomorphism. Hence $\operatorname{Im} j$ is a subgroup of order $[G(X) : 1] = \alpha_0(X^*)$ of $G(X^*)$, and acts transitively on $V(X^*)$. Hence by Lemma 4, $X^* \cong X_{G(X), H}$, where $H = \{\psi \in G(X) \mid V(Y) \cap V(\psi Y) \neq \square\}$. Since X^* is connected, H is a set of generators of $G(X)$.

For $\psi \in H$ define $P_\psi = V(Y) \cap V(\psi Y)$, $Q_\psi = V(Y) \cap V(\psi^{-1} Y)$, and $p_j^{(\psi)} = \psi q_j^{(\psi)}$, $j = 1, \cdots, n_\psi$. Then clearly X is isomorphic to the generalized color-group of $G(X)$ with respect to H and Y.

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