ON NILSTABLE ALGEBRAS

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1. **Introduction.** A simple commutative power-associative algebra $\mathfrak{A}$ of degree 2 over a field $\mathfrak{F}$ of characteristic not 2 has a unity element $1=u+v$ where u and v are orthogonal idempotents. Then $\mathfrak{A}$ may be decomposed relative to u and written as $\mathfrak{A}=\mathfrak{A}_1+\mathfrak{A}_{12}+\mathfrak{A}_2$ with $\mathfrak{A}_1=\mathfrak{A}_u(1)=\mathfrak{A}_v(0)$, $\mathfrak{A}_{12}=\mathfrak{A}_u(1/2)=\mathfrak{A}_v(1/2)$ and $\mathfrak{A}_2=\mathfrak{A}_u(0)=\mathfrak{A}_v(1)$ where x is in $\mathfrak{A}_u(\lambda)$ if and only if $xu=\lambda x$. Furthermore $\mathfrak{A}_1=u\mathfrak{F}+\mathfrak{G}_1$ and $\mathfrak{A}_2=v\mathfrak{F}+\mathfrak{G}_2$ where $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are nilalgebras. It is known that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are orthogonal subalgebras of $\mathfrak{A}$, $\mathfrak{A}_{12}^2\subseteq\mathfrak{A}_1+\mathfrak{A}_2$, and $\mathfrak{A}_{12}\mathfrak{A}_i\subseteq\mathfrak{A}_{12}+\mathfrak{A}_{3-i}$ for $i=1, 2$.[1] Albert has defined u to be a stable idempotent and $\mathfrak{A}$ to be u-stable in case $\mathfrak{A}_{12}\mathfrak{A}_i\subseteq\mathfrak{A}_{12}$ for $i=1, 2$. We generalize this notion and call u a *nilstable idempotent* and $\mathfrak{A}$ *nilstable with respect to* u if $\mathfrak{A}_{12}\mathfrak{A}_i\subseteq\mathfrak{A}_{12}+\mathfrak{G}_{3-i}$ for $i=1, 2$. Thus every stable idempotent is also nilstable. It is known that every commutative power-associative algebra of degree 2 and characteristic 0 is nilstable with respect to every idempotent.[2]

The purpose of this note is to give the proof of the following theorem.

THEOREM 1. *Let $\mathfrak{A}$ be a simple commutative power-associative algebra of degree 2 over a field $\mathfrak{F}$ whose characteristic is prime to 6. Then $\mathfrak{A}$ is a Jordan algebra if and only if $\mathfrak{A}$ is nilstable with respect to two idempotents u, f such that $u\neq 1$, $f\neq 1$, $u+f\neq 1$ and such that f is not of the form $f=u+w_{12}+w_1+w_2$ or $f=v+w_{12}+w_1+w_2$ with w_{12} in $\mathfrak{A}_{12}$, w_1 in $\mathfrak{G}_1$, w_2 in $\mathfrak{G}_2$.*

Since any Jordan algebra is stable with respect to each of its idempotents, we are only concerned with the other half of the theorem. The proof is to a large extent the proof in [8] of the result that every simple commutative power-associative algebra of degree 2 and characteristic 0 is a Jordan algebra. Since we shall lean rather heavily on [8], we shall refer the reader to that paper instead of repeating those results here.

The simple u-stable algebras have been determined by Albert [3; 4; 5]. There remains the problem of finding all algebras which

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[1] The above results are given in [1, 2, and 7].

[2] See [7].

are not u-stable. It is hoped that Theorem 1 will be useful in solving the intermediate problem of determining the nilstable algebras.

2. **Idempotents.** Let $\mathfrak{A}$ be nilstable with respect to an idempotent u. Then if $\mathfrak{F}$ has characteristic not 2 or 3, all the results of §2 of [**8**] are valid here. This is because the assumption of nilstability is the assumption of the conclusion of Lemma 1 of [**8**]. Characteristic not 2 is needed from the outset and it is necessary to divide by 3 at the end of the proof of Lemma 3.

A result needed to prove Theorem 1 is

THEOREM 2. *Let $\mathfrak{A}$ be a commutative power-associative algebra of degree 2 and let u be a nilstable idempotent of $\mathfrak{A}$. If f is any idempotent of $\mathfrak{A}$ other than u, v, 1, then $f=1/2(1+w)$ where $w^2=1$ and $w=\gamma(u-v)+w_{12}+w_1+w_2$ where $w_{12}\neq 0$ is in $\mathfrak{A}_{12}$, w_i is in $\mathfrak{S}_i$, $i=1, 2$.*

If f is any idempotent, then $w=2f-1$ has the property $w^2=1$. Then $w=\gamma u+\delta v+w_{12}+w_1+w_2$ where γ, δ are in $\mathfrak{F}$, w_{12} in $\mathfrak{A}_{12}$, w_1 in $\mathfrak{S}_1$, w_2 in $\mathfrak{S}_2$. When $w_{12}=0$, $w^2=1=\gamma^2u+\delta^2v+2\gamma w_1+2\delta w_2+w_1^2+w_2^2$. Consequently, $\gamma^2=\delta^2=1$, $2\gamma w_1+w_1^2=0$, $2\delta w_2+w_2^2=0$. Since $\gamma\neq 0$, $w_1^2=-2\gamma w_1$ implies $(-(1/2\gamma)w_1)^2=-(1/2\gamma)w_1$ so $-(1/2\gamma)w_1$ is idempotent or zero. But w_1 is nilpotent so $w_1=0$. Similarly, $w_2=0$. Thus $w=\gamma u+\delta v$ where $\gamma=\pm 1$, $\delta=\pm 1$. It follows that $w=1$, -1, $u-v$, or $v-u$, and $f=1/2(1+w)=0$, 1, u, or v. By hypothesis f is not 0, 1, u, or v, so w_{12} must be nonzero.

Computing $w^2=1$ we have $\gamma^2u+\delta^2v+(\gamma+\delta)w_{12}+w_{12}^2+2w_{12}(w_1+w_2)+2\gamma w_1+2\delta w_2+w_1^2+w_2^2=1$. Equating components in $\mathfrak{A}_{12}$, $(\gamma+\delta)w_{12}+2w_{12}[S_{1/2}(w_1)+T_{1/2}(w_2)]=0$ where $w_{12}[S_{1/2}(w_1)+T_{1/2}(w_2)]$ is the component of $w_{12}(w_1+w_2)$ in $\mathfrak{A}_{12}$. It is known [**2**, page 517; and **7**] that $S_{1/2}(w_1)+T_{1/2}(w_2)$ is a nilpotent mapping. Since $w_{12}\neq 0$, it follows that $\gamma+\delta=0$, as desired.

3. **Proof of Theorem 1.** By hypothesis $\mathfrak{A}$ is nilstable with respect to u and f, and, by Theorem 2, $f=1/2(1+w)$, $w=\gamma(u-v)+w_{12}+w_1+w_2$. If $\gamma=0$, the proof of the theorem of reference [**8**] gives us the result of Theorem 1. In making the induction of §4 of [**8**], we consider only the class of algebras satisfying the hypotheses of Theorem 1. Thus we may now assume $\gamma\neq 0$. Even in this case the proof is patterned after that of reference [**8**]. Lemma 8 of [**8**] holds in our situation.

Next we proceed to derive a result comparable to Lemma 9 of [**8**]. Any element of $\mathfrak{A}_f(\lambda)$ has the form $a=\alpha u+\beta v+a_{12}+a_1+a_2$ where α, β are in $\mathfrak{F}$, a_1, a_2 are in $\mathfrak{S}=\mathfrak{S}_1\oplus\mathfrak{S}_2$, a_{12} in $\mathfrak{A}_{12}$. Then $wa=\alpha\gamma u-\beta\gamma v+\gamma(a_1-a_2)+1/2(\alpha+\beta)w_{12}+w_{12}a_{12}+w_{12}(a_1+a_2)+\alpha w_1+\beta w_2$

$+ a_{12}(w_1 + w_2) + a_1w_1 + a_2w_2 = (2\lambda - 1)(\alpha u + \beta v + a_{12} + a_1 + a_2)$. If $\lambda=1$, Lemma 10 of [2] implies $w_{12}a_{12}\equiv\alpha(1-\gamma)=\beta(1+\gamma)$ where by $w_{12}a_{12}\equiv\alpha(1-\gamma)$ we mean $w_{12}a_{12}-\alpha(1-\gamma)1$ is in $\mathfrak{S}$. From $\alpha(1-\gamma)$ $=\beta(1+\gamma)$ we get $2\alpha=\alpha+\alpha\gamma+\beta(1+\gamma)=(\alpha+\beta)(1+\gamma)$ or $\alpha(1-\gamma)$ $=(\alpha+\beta)(1-\gamma^2)/2\equiv w_{12}a_{12}$. Also, $2^{-1}(\alpha+\beta)w_{12}+w_{12}[S_{1/2}(a_1)+T_{1/2}(a_2)]$ $+a_{12}[S_{1/2}(w_1)+T_{1/2}(w_2)]=a_{12}$. Multiply both sides by $w_{12}T_{1/2}(g_2)$, use the fact $w_{12}[S_{1/2}(w_1)+T_{1/2}(w_2)]=0$ which follows from our computation of $w^2=1$, and use the results of [8, §2]. Thus $a_{12}\cdot w_{12}T_{1/2}(g_2)$ is in $\mathfrak{S}$ for any g_2 in $\mathfrak{S}_2$. If $\lambda=0$, the calculations yield $w_{12}a_{12}\equiv -2^{-1}(\alpha+\beta)$ $\cdot(1-\gamma^2)=-\alpha(1+\gamma)=-\beta(1-\gamma)$. When $\lambda=1/2$, $w_{12}a_{12}\equiv-\alpha\gamma=\beta\gamma$ so $\gamma(\alpha+\beta)=0$ and since $\gamma\neq 0$, $\beta=-\alpha$. Some of these results are formally stated in the following lemma.

LEMMA 1. *Any element a of $\mathfrak{A}$ may be written $a=\alpha u+\beta v+a_{12}+a_1+a_2$ with α, β in $\mathfrak{F}$, a_{12} in $\mathfrak{A}_{12}$, a_1, a_2 in $\mathfrak{S}_1$, $\mathfrak{S}_2$, respectively. If a is in $\mathfrak{A}_f(\lambda)$, $\lambda=0$, 1, then $a_{12}\cdot w_{12}T_{1/2}(g_2)$ is in $\mathfrak{S}$ for any g_2 in $\mathfrak{S}_2$. When a is in $\mathfrak{A}_f(1/2)$, $\beta=-\alpha$.*

Now let a be in $\mathfrak{A}_f(1)$ and let $b=\zeta u+\eta v+b_{12}+b_1+b_2$ also be in $\mathfrak{A}_f(1)$. Then $ab = \alpha\zeta u + \beta\eta v + 2^{-1}(\alpha + \beta)b_{12} + 2^{-1}(\zeta + \eta)a_{12} + a_{12}b_{12}$ $+ a_{12}(b_1 + b_2) + b_{12}(a_1 + a_2) + \alpha b_1 + \beta b_2 + \zeta a_1 + \eta a_2 + a_1b_1 + a_2b_2$ and ab is in $\mathfrak{A}_f(1)$. Let $ab=\theta u+\phi v+c_{12}+c_1+c_2$ and let $a_{12}b_{12}\equiv\rho$. We have $\theta=\alpha\zeta+\rho$, $\phi=\beta\eta+\rho$. From the results above Lemma 1, we have $(\alpha\zeta+\rho)(1-\gamma)=(\beta\eta+\rho)(1+\gamma)$. Also, $\zeta(1-\gamma)=\eta(1+\gamma)$, $\beta(1+\gamma)$ $=\alpha(1-\gamma)$. Therefore, $\alpha\eta(1+\gamma)+\rho(1-\gamma)=\alpha\eta(1-\gamma)+\rho(1+\gamma)$, and and $2\alpha\eta\gamma=2\rho\gamma$. Since $\gamma\neq 0$, $\rho=\alpha\eta$ and $\theta=\alpha(\zeta+\eta)$, $\phi=\eta(\alpha+\beta)$.

LEMMA 2. *Let $a=\alpha u+\beta v+a_{12}+a_1+a_2$ and $b=\zeta u+\eta v+b_{12}+b_1+b_2$ be any two elements of $\mathfrak{A}_f(1)$. Then if $\gamma\neq 0$, $a_{12}b_{12}\equiv\alpha\eta$ and $ab=\alpha(\zeta+\eta)u$ $+\eta(\alpha+\beta)v+c_{12}+c_1+c_2$.*

COROLLARY 1. *Let $a=\alpha u+\beta v+a_{12}+a_1+a_2$ be any element of $\mathfrak{A}_f$ (1). Then $a^k=\alpha(\alpha+\beta)^{k-1}u+\beta(\alpha+\beta)^{k-1}v+c_{12}+c_1+c_2$.*

LEMMA 3. *If a is nilpotent, then $\alpha=\beta=0$.*

The corollary to Lemma 2 implies that if a is nilpotent $\alpha=0$, $\beta=0$ or $\alpha+\beta=0$. If $\beta=-\alpha$, the fact that $\alpha(1-\gamma)=\beta(1+\gamma)$ implies $\alpha=\beta=0$.

The results of Lemma 2, its corollary, and Lemma 3 can be proved in the same manner for elements of $\mathfrak{A}_f(0)$.

LEMMA 4. *Let $a=\alpha u+\beta v+a_{12}+a_1+a_2$ be any element in $\mathfrak{A}_f(1)$ or $\mathfrak{A}_f(0)$ and $c=\delta(u-v)+c_{12}+c_1+c_2$ be any element of $\mathfrak{A}_f(1/2)$. Then $a_{12}c_{12}\equiv 2^{-1}(\beta-\alpha)\delta$. If a is nilpotent, $a_{12}c_{12}\equiv 0$.*

The product $ac=m+n$ where m is in $\mathfrak{A}_f(1/2)$ and n is in $\mathfrak{A}_f(1-\lambda)$ when a is in $\mathfrak{A}_f(\lambda)$, $(\lambda=0, 1)$. By assumption $\mathfrak{A}$ is nilstable with respect to f so n is nilpotent. Lemma 3 and the corresponding result for elements in $\mathfrak{A}_f(0)$ imply $n=n_{12}+n_1+n_2$ with n_1, n_2 in $\mathfrak{G}$. By Lemma 1, $m=\mu(u-v)+m_{12}+m_1+m_2$. From the equation $ac=m+n$ we obtain $\alpha\delta u - \beta\delta v + a_{12}c_{12} \equiv \mu(u-v)$ and it follows that $\mu-\alpha\delta = -\mu+\beta\delta$, $\mu=2^{-1}(\alpha+\beta)\delta$, $a_{12}c_{12}\equiv 2^{-1}(\beta-\alpha)\delta$.

As in [8], consider any element g_2 in $\mathfrak{G}_2$ and write $g_2=g_f(1)+g_f(0)+g_f(1/2)$ where $g_f(\lambda)$ is in $\mathfrak{A}_f(\lambda)$. Write each $g_f(\lambda)$ as a sum of elements determined by the decomposition of $\mathfrak{A}$ relative to u. Let $g_f(1)=\alpha u+\beta v+a_{12}+a_1+a_2$, $g_f(0)=\zeta u+\eta v+b_{12}+b_1+b_2$, $g_f(1/2)=\phi(u-v)+d_{12}+d_1+d_2$. Then $\alpha+\zeta=-\phi=-\beta-\eta$. The results before Lemma 1 imply $\alpha(1-\gamma)=\beta(1+\gamma)$ and $\zeta(1+\gamma)=\eta(1-\gamma)$. Subtract the second from the first of these relations to get $\alpha-\zeta=\beta-\eta$. Add corresponding sides of this relation to $\alpha+\zeta=-\beta-\eta$ so that $\alpha=-\eta$, and then $\beta=-\zeta$. Now by Lemma 4 and for any c_{12} belonging to $\mathfrak{A}_f(1/2)$, $(a_{12}-b_{12})c_{12}\equiv 2^{-1}(\beta-\alpha)\delta-2^{-1}(\eta-\zeta)\delta=0$. By Lemma 8 of [8], $a_{12}-b_{12}=w_{12}T_{1/2}(g_2)$. Thus we have proved Lemma 12 and Theorem 1 of [8].

The induction of §4 of [8] made in the class of algebras nilstable with respect to two idempotents u, f such that $u\neq 1$, $f\neq 1$, $u+f\neq 1$, $f\neq u+z_{12}+z_1+z_2$, and $f\neq v+z_{12}+z_1+z_2$ with z_{12} in $\mathfrak{A}_{12}$, z_1 in $\mathfrak{A}_1$, z_2 in $\mathfrak{A}_2$, and the induction completes the proof of Theorem 1. In order to successfully complete the induction it is necessary to have w_{12}^2 nonnilpotent. Since $w_{12}^2\equiv(1-\gamma^2)$, this means we need to have $\gamma^2-1\neq 0$, $\gamma\neq\pm 1$. The condition $\gamma=\pm 1$ implies

$$f = (1 + w)/2 = u + 2^{-1}(w_{12} + w_1 + w_2)$$

or $f=v+2^{-1}(w_{12}+w_1+w_2)$.

4. **The cases $\gamma=\pm 1$.** The result of Theorem 1 is not true when $\gamma=\pm 1$. For example consider the u-stable algebra $\mathfrak{S}$ of characteristic $p>5$ described in [6]. It is not a Jordan algebra and $\mathfrak{S}=u\mathfrak{A}+v\mathfrak{A}+y_0\mathfrak{A}+y_1\mathfrak{A}$ where $\mathfrak{A}=\mathfrak{F}[1, x]$, $x^p=0$. In the decomposition relative to u, $\mathfrak{S}_{12}=y_0\mathfrak{A}+y_1\mathfrak{A}$, $\mathfrak{S}_1=u\mathfrak{A}$, $\mathfrak{S}_2=v\mathfrak{A}$. Let $w=u-v+2y_1x^{p-1}$ so that $f=2^{-1}(1+w)=u+y_1x^{p-1}$. If a is in $\mathfrak{S}_f(1)$, $a=\alpha u+\beta v+a_{12}+a_1+a_2$. The proof of Lemma 1 implies $\alpha(1-\gamma)=\beta(1+\gamma)$. Since $\gamma=1$, $\beta=0$. Furthermore $wa=a$, $\alpha u+a_1-a_2+\alpha y_1x^{p-1}+2y_1x^{p-1}\cdot a_{12}=\alpha u+a_{12}+a_1+a_2$ where we have used the fact that $y_1x^{p-1}(a_1+a_2)=0$. It follows that $a_{12}=\alpha y_1x^{p-1}$ and hence $a_2=0$. Thus $a=\alpha u+\alpha y_1x^{p-1}+a_1=\alpha f+a_1$ where a_1 is any nilpotent element of $\mathfrak{S}_1$. Similarly, if b is any element of $\mathfrak{S}_f(0)$, $b=\beta v-\beta y_1x^{p-1}+b_2=\beta(1-f)+b_2$ where b_2 is any nilpotent

element of $\mathfrak{S}_2$. Next let c be any element of $\mathfrak{S}_f(1/2)$. By Lemma 1, $c=\lambda(u-v)+c_{12}+c_1+c_2$ and $wc=0$. The product $wc=0$ implies $\lambda(u+v)+c_1-c_2+2y_1x^{p-1}\cdot c_{12}=0$. The multiplication table of $\mathfrak{S}$ implies $2y_1x^{p-1}\cdot c_{12}$ is nilpotent and therefore it is equal to c_2-c_1 and $\lambda=0$. The product $ca=(c_{12}+c_1+c_2)(\alpha f+a_1)=2^{-1}\alpha(c_{12}+c_1+c_2)+(c_{12}+c_1)a_1$. We know ca is in $\mathfrak{S}_f(1/2)+\mathfrak{S}_f(0)$, so $w(ca)$ is the negative of the component in $\mathfrak{S}_f(0)$. Computing, we get $w(ca)=2y_1x^{p-1}\cdot c_{12}a_1+c_1a_1$. By the definition of multiplication in $\mathfrak{S}$, $y_1x^{p-1}\cdot c_{12}a_1$ is 0 or a scalar multiple of x^{p-1}. Since $2y_1x^{p-1}\cdot c_{12}=c_2-c_1$, $c_1a_1=-(2y_1x^{p-1}\cdot c_{12})a_1$, which is a scalar multiple of $x^{p-1}u$. Therefore, the component of ca in $\mathfrak{S}_f(0)$ is nilpotent. Similarly, cb is the sum of an element in $\mathfrak{S}_f(1/2)$ and a nilpotent element of $\mathfrak{S}_f(1)$. This proves that $\mathfrak{S}$ is nilstable with respect to f, and shows that the restriction $\gamma\neq\pm 1$ is necessary in order to obtain the result of Theorem 1. In our example we took $\gamma=1$, but we could just as easily let $\gamma=-1$, $w=v-u+2y_1x^{p-1}$, and $f=v+y_1x^{p-1}$.

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