ON NILSTABLE ALGEBRAS

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1. Introduction. A simple commutative power-associative algebra \( \mathcal{A} \) of degree 2 over a field \( \mathbb{F} \) of characteristic not 2 has a unity element \( 1 = u + v \) where \( u \) and \( v \) are orthogonal idempotents. Then \( \mathcal{A} \) may be decomposed relative to \( u \) and written as \( \mathcal{A} = \mathcal{A}_1 + \mathcal{A}_{12} + \mathcal{A}_2 \) with \( \mathcal{A}_1 = \mathcal{A}_u(1) = \mathcal{A}_v(0), \mathcal{A}_{12} = \mathcal{A}_u(1/2) = \mathcal{A}_v(1/2) \) and \( \mathcal{A}_2 = \mathcal{A}_u(0) = \mathcal{A}_v(1) \) where \( x \) is in \( \mathcal{A}_x(\lambda) \) if and only if \( xu = \lambda x \). Furthermore \( \mathcal{A}_1 = u\mathcal{F} + \mathcal{G}_1 \) and \( \mathcal{A}_2 = v\mathcal{F} + \mathcal{G}_2 \) where \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are nilalgebras. It is known that \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are orthogonal subalgebras of \( \mathcal{A} \), \( \mathcal{A}_{12} \subseteq \mathcal{A}_1 + \mathcal{A}_2 \), and \( \mathcal{A}_{12} \mathcal{A}_i \subseteq \mathcal{A}_{12} + \mathcal{A}_{3-i} \) for \( i = 1, 2 \). Albert has defined \( u \) to be a stable idempotent and \( \mathcal{A} \) to be \( u \)-stable in case \( \mathcal{A}_{12} \mathcal{A}_i \subseteq \mathcal{A}_i \) for \( i = 1, 2 \). We generalize this notion and call \( u \) a nilstable idempotent and \( \mathcal{A} \) nilstable with respect to \( u \) if \( \mathcal{A}_{12} \mathcal{A}_i \subseteq \mathcal{A}_i + \mathcal{A}_{3-i} \) for \( i = 1, 2 \). Thus every stable idempotent is also nilstable. It is known that every commutative power-associative algebra of degree 2 and characteristic 0 is nilstable with respect to every idempotent. \(^2\)

The purpose of this note is to give the proof of the following theorem.

**Theorem 1.** Let \( \mathcal{A} \) be a simple commutative power-associative algebra of degree 2 over a field \( \mathbb{F} \) whose characteristic is prime to 6. Then \( \mathcal{A} \) is a Jordan algebra if and only if \( \mathcal{A} \) is nilstable with respect to two idempotents \( u, f \) such that \( u \neq 1, f \neq 1, u + f \neq 1 \) and such that \( f \) is not of the form \( f = u + w_{12} + w_1 + w_2 \) or \( f = v + w_{12} + w_1 + w_2 \) with \( w_{12} \) in \( \mathcal{A}_{12} \), \( w_1 \) in \( \mathcal{G}_1 \), \( w_2 \) in \( \mathcal{G}_2 \).

Since any Jordan algebra is stable with respect to each of its idempotents, we are only concerned with the other half of the theorem. The proof is to a large extent the proof in [8] of the result that every simple commutative power-associative algebra of degree 2 and characteristic 0 is a Jordan algebra. Since we shall lean rather heavily on [8], we shall refer the reader to that paper instead of repeating those results here.

The simple \( u \)-stable algebras have been determined by Albert [3; 4; 5]. There remains the problem of finding all algebras which

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\(^2\) The above results are given in [1, 2, and 7].

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are not $u$-stable. It is hoped that Theorem 1 will be useful in solving the intermediate problem of determining the nilstable algebras.

2. Idempotents. Let $\mathfrak{A}$ be nilstable with respect to an idempotent $u$. Then if $\mathfrak{A}$ has characteristic not 2 or 3, all the results of §2 of [8] are valid here. This is because the assumption of nilstability is the assumption of the conclusion of Lemma 1 of [8]. Characteristic not 2 is needed from the outset and it is necessary to divide by 3 at the end of the proof of Lemma 3.

A result needed to prove Theorem 1 is

Theorem 2. Let $\mathfrak{A}$ be a commutative power-associative algebra of degree 2 and let $u$ be a nilstable idempotent of $\mathfrak{A}$. If $f$ is any idempotent of $\mathfrak{A}$ other than $u$, $v$, 1, then $f = 1/2(1 + w)$ where $w^2 = 1$ and $w = \gamma(u - v)$ + $w_1 + w_2$ where $w_1 \neq 0$ is in $\mathfrak{A}_2$, $w_i$ is in $\mathfrak{O}_i$, $i = 1, 2$.

If $f$ is any idempotent, then $w = 2f - 1$ has the property $w^2 = 1$. Then $w = \gamma u + \delta v + w_1 + w_2$ where $\gamma, \delta$ are in $\mathfrak{O}$, $w_1$ in $\mathfrak{A}_2$, $w_2$ in $\mathfrak{O}_2$. When $w_1 = 0$, $w^2 = 1 = \gamma^2 u + \delta^2 v + 2\gamma w_1 + 2\delta w_2 + w_1^2 + w_2^2$. Consequently, $\gamma^2 = \delta^2 = 1$, $2\gamma w_1 + w_2^2 = 0$, $2\delta w_2 + w_1^2 = 0$. Since $\gamma \neq 0$, $w_1^2 = -2\gamma w_1$ implies $-(1/2\gamma) w_1^2 = -(1/2\gamma) w_1$ so $-(1/2\gamma) w_1$ is idempotent or zero. But $w_1$ is nilpotent so $w_1 = 0$. Similarly, $w_2 = 0$. Thus $w = \gamma u + \delta v$ where $\gamma = \pm 1$, $\delta = \pm 1$. It follows that $w = 1, -1, u - v$, or $v - u$, and $f = 1/2(1 + w) = 0, 1, u$, or $v$. By hypothesis $f$ is not 0, 1, $u$, or $v$, so $w_1$ must be nonzero.

Computing $w^2 = 1$ we have $\gamma^2 u + \delta^2 v + (\gamma + \delta) w_1 + w_2^2 + 2w_1 (w_1 + w_2) + 2\gamma w_1 + 2\delta w_2 + w_1^2 + w_2^2 = 1$. Equating components in $\mathfrak{A}_2$, $(\gamma + \delta) w_1 + 2w_1 [S_{1/2}(w_1) + T_{1/2}(w_2)] = 0$ where $w_1 [S_{1/2}(w_1) + T_{1/2}(w_2)]$ is the component of $w_1 (w_1 + w_2)$ in $\mathfrak{A}_2$. It is known [2, page 517; and 7] that $S_{1/2}(w_1) + T_{1/2}(w_2)$ is a nilpotent mapping. Since $w_1 \neq 0$, it follows that $\gamma + \delta = 0$, as desired.

3. Proof of Theorem 1. By hypothesis $\mathfrak{A}$ is nilstable with respect to $u$ and $f$, and, by Theorem 2, $f = 1/2(1 + w)$, $w = \gamma(u - v) + w_1 + w_2$. If $\gamma = 0$, the proof of the theorem of reference [8] gives us the result of Theorem 1. In making the induction of §4 of [8], we consider only the class of algebras satisfying the hypotheses of Theorem 1. Thus we may now assume $\gamma \neq 0$. Even in this case the proof is patterned after that of reference [8]. Lemma 8 of [8] holds in our situation.

Next we proceed to derive a result comparable to Lemma 9 of [8]. Any element of $\mathfrak{A}(\lambda)$ has the form $a = \alpha u + \beta v + a_1 + a_2$ where $\alpha, \beta$ are in $\mathfrak{O}$, $a_1, a_2$ are in $\mathfrak{O} = \mathfrak{O}_1 \oplus \mathfrak{O}_2$, $a_12$ in $\mathfrak{A}_1$. Then $wa = \alpha \gamma u - \beta \gamma v + \gamma(a_1 - a_2) + 1/2(\alpha + \beta) w_{12} + w_{12} a_{12} + w_{12} (a_1 + a_2) + \alpha w_1 + \beta w_2$
+ a_{12}(w_1 + w_2) + a_1w_1 + a_2w_2 = (2\lambda - 1)(\alpha u + \beta v + a_{12} + a_1 + a_2).

If \lambda = 1, Lemma 10 of [2] implies $w_{12}a_{12} = \alpha(1 - \gamma) = \beta(1 + \gamma)$ where by $w_{12}a_{12} = \alpha(1 - \gamma)$ we mean $w_{12}a_{12} - \alpha(1 - \gamma)1$ is in $\mathfrak{g}$. From $\alpha(i - \gamma)$ $= \beta(1 + \gamma)$ we get $2\alpha = \alpha + \alpha\gamma + \beta(1 + \gamma) = (\alpha + \beta)(1 + \gamma)$ or $\alpha(1 - \gamma) = (\alpha + \beta)(1 - \gamma^2)/2 \equiv w_{12}a_{12}$. Also, $2^{-1}(\alpha + \beta)w_{12} + w_{12}[S_{1/2}(a_1) + T_{1/2}(a_2)] + a_{12}[S_{1/2}(w_1) + T_{1/2}(w_2)] = a_{12}$. Multiply both sides by $w_{12}T_{1/2}(g_2)$, use the fact $w_{12}[S_{1/2}(w_1) + T_{1/2}(w_2)] = 0$ which follows from our computation of $w^2 = 1$, and use the results of [8, §2]. Thus $a_{12} \cdot w_{12}T_{1/2}(g_2)$ is in $\mathfrak{g}$ for any $g_2$ in $\mathfrak{g}_2$. If $\lambda = 0$, the calculations yield $w_{12}a_{12} \equiv -2^{-1}(\alpha + \beta) \cdot (1 - \gamma^2) = -\alpha(1 + \gamma) = -\beta(1 - \gamma)$. When $\lambda = 1/2$, $w_{12}a_{12} \equiv -\alpha\gamma = \beta\gamma$ so $\gamma(\alpha + \beta) = 0$ and since $\gamma \neq 0$, $\beta = -\alpha$. Some of these results are formally stated in the following lemma.

**Lemma 1.** Any element $a$ of $\mathfrak{g}$ may be written $a = \alpha u + \beta v + a_{12} + a_1 + a_2$ with $\alpha, \beta$ in $F$, $a_{12} \in \mathfrak{g}_{12}$, $a_1, a_2$ in $\mathfrak{g}_1, \mathfrak{g}_2$, respectively. If $a$ is in $\mathfrak{g}_f(\lambda)$, $\lambda = 0, 1$, then $a_{12} \cdot w_{12}T_{1/2}(g_2)$ is in $\mathfrak{g}$ for any $g_2$ in $\mathfrak{g}_2$. When $a$ is in $\mathfrak{g}_f(1/2)$, $\beta = -\alpha$.

Now let $a$ be in $\mathfrak{g}_f(1)$ and let $b = \xi u + \eta v + b_{12} + b_1 + b_2$ also be in $\mathfrak{g}_f(1)$. Then $ab = \alpha\xi u + \beta\eta v + 2^{-1}(\alpha + \beta)b_{12} + 2^{-1}(\xi + \eta)a_{12} + a_{12}(b_1 + b_2) + b_1(a_1 + a_2) + a_2(a_1 + a_2) + \alpha b_2 + \beta a_1 + \eta a_2 + a_1 b_1 + a_2 b_2$ and $ab$ is in $\mathfrak{g}_f(1)$. Let $ab = \theta u + \phi v + c_1 + c_2$ and let $a_{12}b_{12} \equiv \rho$. We have $\theta = \alpha\xi + \rho$, $\phi = \beta \eta + \rho$. From the results above Lemma 1, we have $(\alpha\xi + \rho)(1 - \gamma) = (\beta \eta + \rho)(1 + \gamma)$. Also, $(\xi(1 - \gamma) = \eta(1 + \gamma)$, $\beta(1 + \gamma) = \alpha(1 - \gamma)$. Therefore, $\alpha \eta(1 + \gamma) + \rho(1 - \gamma) = \alpha \eta(1 - \gamma) + \rho(1 + \gamma)$, and $2\alpha \eta \gamma = 2\rho \gamma$. Since $\gamma \neq 0$, $\rho = \alpha \eta$ and $\theta = \alpha(\xi + \eta)$, $\phi = \eta(\alpha + \beta)$.

**Lemma 2.** Let $a = \alpha u + \beta v + a_{12} + a_1 + a_2$ and $b = \xi u + \eta v + b_{12} + b_1 + b_2$ be any two elements of $\mathfrak{g}_f(1)$. Then if $\gamma \neq 0$, $a_{12}b_{12} \equiv \alpha \eta$ and $ab = \alpha(\xi + \eta)u + \eta(\alpha + \beta)v + c_1 + c_2$.

**Corollary 1.** Let $a = \alpha u + \beta v + a_{12} + a_1 + a_2$ be any element of $\mathfrak{g}_f(1)$. Then $a^k = \alpha(\alpha + \beta)^{k-1}u + \beta(\alpha + \beta)^{k-1}v + c_1 + c_2$.

**Lemma 3.** If $a$ is nilpotent, then $\alpha = \beta = 0$.

The corollary to Lemma 2 implies that if $a$ is nilpotent $\alpha = 0$, $\beta = 0$ or $\alpha + \beta = 0$. If $\beta = -\alpha$, the fact that $\alpha(1 - \gamma) = \beta(1 + \gamma)$ implies $\alpha = \beta = 0$.

The results of Lemma 2, its corollary, and Lemma 3 can be proved in the same manner for elements of $\mathfrak{g}_f(0)$.

**Lemma 4.** Let $a = \alpha u + \beta v + a_{12} + a_1 + a_2$ be any element in $\mathfrak{g}_f(1)$ or $\mathfrak{g}_f(0)$ and $c = \delta(u - v) + c_1 + c_2$ be any element of $\mathfrak{g}_f(1/2)$. Then $a_{12}c_{12} \equiv 2^{-1}(\beta - \alpha)\delta$. If $a$ is nilpotent, $a_{12}c_{12} \equiv 0$. 
The product $ac = m + n$ where $m$ is in $\mathfrak{A}_f(1/2)$ and $n$ is in $\mathfrak{A}_f(1 - \lambda)$ when $a$ is in $\mathfrak{A}_f(\lambda)$, $(\lambda = 0, 1)$. By assumption $\mathfrak{A}$ is nilstable with respect to $f$ so $n$ is nilpotent. Lemma 3 and the corresponding result for elements in $\mathfrak{A}_f(0)$ imply $n = n_{12} + n_1 + n_2$ with $n_1, n_2$ in $\mathfrak{O}$. By Lemma 1, $m = \mu(u - v) + m_{12} + m_1 + m_2$. From the equation $ac = m + n$ we obtain $\alpha \delta u - \beta \delta v + a_{12} c_{12} = \mu(u - v)$ and it follows that $\mu - \alpha \delta = -\mu + \beta \delta$, $\mu = 2^{-1}(\alpha + \beta) \delta$, $a_{12} c_{12} = 2^{-1}(\beta - \alpha) \delta$.

As in [8], consider any element $g_2$ in $\mathfrak{O}_2$ and write $g_2 = g_f(1) + g_f(0) + g_f(1/2)$ where $g_f(\lambda)$ is in $\mathfrak{A}_f(\lambda)$. Write each $g_f(\lambda)$ as a sum of elements determined by the decomposition of $\mathfrak{A}$ relative to $u$. Let $g_f(1) = \alpha u + \beta v + a_{12} + a_1 + a_2$, $g_f(0) = \xi u + \eta v + b_{12} + b_1 + b_2$, $g_f(1/2) = \phi(u - v) + d_{12} + d_1 + d_2$. Then $\alpha + \xi = -\phi = -\beta - \eta$. The results before Lemma 1 imply $\alpha(1 - \gamma) = \beta(1 + \gamma)$ and $\xi(1 + \gamma) = \eta(1 - \gamma)$. Subtract the second from the first of these relations to get $\alpha - \xi = \beta - \eta$. Add corresponding sides of this relation to $\alpha + \xi = -\beta - \eta$ so that $\alpha = -\eta$, and then $\beta = -\xi$. Now by Lemma 4 and for any $c_{12}$ belonging to $\mathfrak{A}_f(1/2)$, $(a_{12} - b_{12}) c_{12} = 2^{-1}(\beta - \alpha) \delta - 2^{-1}(\eta - \xi) \delta = 0$. By Lemma 8 of [8], $a_{12} - b_{12} = w_{12} T_{1/2}(g_2)$. Thus we have proved Lemma 12 and Theorem 1 of [8].

The induction of §4 of [8] made in the class of algebras nilstable with respect to two idempotents $u, f$ such that $u \neq 1, f \neq 1, u + f \neq 1$, $f \neq u + z_{12} + z_1 + z_2$, and $f \neq v + z_{12} + z_1 + z_2$ with $z_{12}$ in $\mathfrak{A}_{12}$, $z_1$ in $\mathfrak{A}_1$, $z_2$ in $\mathfrak{A}_2$, and the induction completes the proof of Theorem 1. In order to successfully complete the induction it is necessary to have $w_{12}^2$ non-nilpotent. Since $w_{12}^2 \equiv (1 - \gamma^2)$, this means we need to have $\gamma^2 - 1 \neq 0$, $\gamma \neq \pm 1$. The condition $\gamma = \pm 1$ implies

$$f = (1 + w)/2 = u + 2^{-1}(w_{12} + w_1 + w_2)$$

or $f = v + 2^{-1}(w_{12} + w_1 + w_2)$.

4. The cases $\gamma = \pm 1$. The result of Theorem 1 is not true when $\gamma = \pm 1$. For example consider the $u$-stable algebra $\mathfrak{O}$ of characteristic $p > 5$ described in [6]. It is not a Jordan algebra and $\mathfrak{O} = u \mathfrak{A} + v \mathfrak{A} + y \mathfrak{A} + y \mathfrak{A}$ where $\mathfrak{A} = \mathfrak{H}[1, x]$, $x^p = 0$. In the decomposition relative to $u$, $\mathfrak{O}_{12} = y \mathfrak{A} + y \mathfrak{A}$, $\mathfrak{O}_1 = u \mathfrak{A}$, $\mathfrak{O}_2 = v \mathfrak{A}$. Let $w = u - v + 2y_1 x^{p-1}$ so that $f = 2^{-1}(1 + w) = u + y_1 x^{p-1}$. If $a$ is in $\mathfrak{O}_f(1)$, $a = \alpha u + \beta v + a_{12} + a_1 + a_2$. The proof of Lemma 1 implies $\alpha(1 - \gamma) = \beta(1 + \gamma)$. Since $\gamma = 1, \beta = 0$. Furthermore $wa = a$, $\alpha u + a_1 - a_2 + \alpha y_1 x^{p-1} + 2y_1 x^{p-1} \cdot a_{12} = \alpha u + a_1 + a_2$ where we have used the fact that $y_1 x^{p-1}(a_1 + a_2) = 0$. It follows that $a_{12} = \alpha y_1 x^{p-1}$ and hence $a_2 = 0$. Thus $a = \alpha u + \alpha y_1 x^{p-1} + a_1 = \alpha f + a_1$ where $a_1$ is any nilpotent element of $\mathfrak{O}$. Similarly, if $b$ is any element of $\mathfrak{O}_f(0)$, $b = \beta v - \beta y_1 x^{p-1} + b_2 = \beta(-f) + b_2$ where $b_2$ is any nilpotent
element of \( \mathfrak{S}_2 \). Next let \( c \) be any element of \( \mathfrak{S}_f(1/2) \). By Lemma 1, 
\[ c = \lambda (u - v) + c_{12} + c_1 + c_2 \quad \text{and} \quad wc = 0. \]

The product \( wc = 0 \) implies \( \lambda (u + v) + c_1 - c_2 + 2y_1x^{p-1} \cdot c_{12} = 0 \). The multiplication table of \( \mathfrak{S} \) implies \( 2y_1x^{p-1} \cdot c_{12} \) is nilpotent and therefore it is equal to \( c_2 - c_1 \) and \( \lambda = 0 \). The product \( ca = (c_{12} + c_1 + c_2)(\alpha f + a_1) = 2^{-1} \alpha (c_{12} + c_1 + c_2) + (c_{12} + c_1)a_1 \). We know \( ca \) is in \( \mathfrak{S}_f(1/2) + \mathfrak{S}_f(0) \), so \( w(ca) \) is the negative of the component in \( \mathfrak{S}_f(0) \). Computing, we get \( w(ca) = 2y_1x^{p-1} \cdot c_{12}a_1 + c_1a_1 \).

By the definition of multiplication in \( \mathfrak{S} \), \( y_1x^{p-1} \cdot c_{12}a_1 \) is 0 or a scalar multiple of \( x^{p-1} \). Since \( 2y_1x^{p-1} \cdot c_{12} = c_2 - c_1 \), \( c_1a_1 = -(2y_1x^{p-1} \cdot c_{12})a_1 \), which is a scalar multiple of \( x^{p-1} \). Therefore, the component of \( ca \) in \( \mathfrak{S}_f(0) \) is nilpotent. Similarly, \( cb \) is the sum of an element in \( \mathfrak{S}_f(1/2) \) and a nilpotent element of \( \mathfrak{S}_f(1) \). This proves that \( \mathfrak{S} \) is nilstable with respect to \( f \), and shows that the restriction \( \gamma \neq \pm 1 \) is necessary in order to obtain the result of Theorem 1. In our example we took \( \gamma = 1 \), but we could just as easily let \( \gamma = -1 \), \( w = v - u + 2y_1x^{p-1} \), and \( f = v + y_1x^{p-1} \).

References


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