ON SPACES WHICH ARE NOT OF COUNTABLE CHARACTER

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It is well known that the unit interval $I$ has a countable base and the fixed point property. By considering the maps $g(x) = x^2$ and $h(x) = 1 - x$, one sees that there is no $x \in I$ such that for every continuous map $f: I \to I$, $x \in f(I)$ implies $f(x) = x$.

In Theorem 1, it is shown that if $A$ is a closed, non-null proper subset of a locally connected, compact Hausdorff space $X$ which has a countable base, then there exists a continuous map $f: X \to X$ such that $A \cap f(X)$ is not contained in $A \cap f(A)$. Theorem 2 shows that certain nondegenerate topological spaces $X$ contain proper subsets $M$ such that for every continuous map $f: X \to X$, $M \cap f(X) \subseteq M \cap f(M)$. That is, for each of these spaces $X$ and every continuous map $f: X \to X$, $x \in M \cap f(X)$ implies $f^{-1}(x) \cap M \neq \emptyset$. The corollary is of interest in that, if $X$ satisfies the hypotheses of Theorem 2 and $M$ consists of a single point, then a fixed point of some of the maps $f: X \to X$ is located.

**Theorem 1.** Suppose $X$ is a connected, locally connected, compact Hausdorff space which has a countable base. If $A$ is any non-null, closed, proper subset of $X$, then there exists a continuous map $f: X \to X$ such that $A \cap f(X) \setminus A \cap f(A) \neq \emptyset$.

**Proof.** Since $X$ is compact Hausdorff and has a countable base, $X$ is metrizable. Hence $X$ is arcwise connected. Let $y \in X \setminus A$. Since

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X is normal, there exists a continuous map \( h \) such that \( h(x) = 0 \) for \( x \in A \), \( h(y) = 1 \), and \( 0 \leq h(x) \leq 1 \) for each \( x \in X \). Since \( X \) is arcwise connected, there is an arc \( C \) connecting \( y \) and \( A \). Now \( C \) contains a subarc \( C_1 \), such that \( y \in C_1 \) and \( C_1 \cap A \) is a single point \( x_0 \). Then there is a homeomorphism \( g \) such that \( g([0, 1]) = C_1 \), \( g(0) = y \), and \( g(1) = x_0 \). Consider the continuous map \( f = gh \). Clearly \( f : X \to X \) and \( x_0 \in A \cap f(X) \).

But since \( x_0 \neq y \) and \( f(A) = y, \ x_0 \notin f(A) \). Hence \( x_0 \in A \cap f(A) \); and \( f = gh \) is the required map.

In the following let \( M \) consist of the set of all points \( x \in X \) such that if \( x \) is a limit point of \( \{ y_n \} \) where \( U_{y_n} \subset X \setminus x \), then \( \{ y_n \} \) contains uncountably many distinct points. It may be noted that \( X \) does not satisfy the first axiom of countability at points of \( M \).

**Theorem 2.** Let \( X \) be a connected Hausdorff space which contains a non-null set \( M \) such that \( M = \overline{M} \), and \( M \neq X \). Suppose also that each point of \( X \setminus M \) has a countable base. Then for every continuous map \( f : X \to X \), \( M \cap f(X) \subset M \cap f(M) \).

**Proof.** Let \( f \) be a continuous function such that \( f \) maps \( X \) into \( X \). If \( M \cap f(X) = \emptyset \), then \( M \cap f(X) \subset M \cap f(M) \). On the other hand, suppose \( x \in M \cap f(X) \) and \( x \notin f(M) \). Now \( x \) is a limit point of \( X \setminus x \), for otherwise \( X \) would not be connected. Since \( x \in f(M) \), \( f^{-1}(x) \subset X \setminus M \). Suppose there exists \( z \in f^{-1}(x) \) such that every neighborhood of \( z \) intersects \( X \setminus f^{-1}(x) \). Since \( X \setminus M \) is open and \( z \in X \setminus M \), there exists a countable set \( \{ U_n(z) \} \) of neighborhoods of \( z \) such that \( \bigcap_{n=1}^{\infty} U_n(z) = z \), and \( U_n(z) \subset X \setminus M \) for each \( n \). In each \( U_n(z) \) there exists a point \( u_n \) such that \( u_n \in X \setminus f^{-1}(x) \). Now \( f(u_n) \subset X \setminus x \) for each \( n \); and, by the continuity of \( f \), \( x \) is a limit point of the set \( \bigcup_{n=1}^{\infty} f(u_n) \). But \( \bigcup_{n=1}^{\infty} f(u_n) \) does not contain uncountably many distinct points. Thus a contradiction has been reached. Suppose that for every \( z \in f^{-1}(x) \), there exists a neighborhood \( U(z) \) such that \( U(z) \cap \{ X \setminus f^{-1}(x) \} = \emptyset \). Clearly \( U(z) \) may be taken so that \( U(z) \subset X \setminus M \). Then \( U(z) \subset f^{-1}(x) \) for every \( z \in f^{-1}(x) \), and \( f^{-1}(x) \) is open in \( X \). Since \( x \) is closed in \( X \), \( f^{-1}(x) \) is closed in \( X \). Therefore \( f^{-1}(x) = X \) and \( f(X) = x \). But \( f(M) \subset f(X) \); hence, \( f(M) = x \). But this contradicts the assumption that \( x \notin f(M) \).

**Corollary.** If \( X \) is a nondegenerate connected Hausdorff space in which \( M \) is a single point \( x_0 \), then for every continuous function \( f \) such that \( f : X \to X \) and \( x_0 \in f(X) \), \( f(x_0) = x_0 \).

**Proof.** By Theorem 2, \( x_0 \subset f(x_0) \). Since \( f(x_0) \) is a single point, \( x_0 = f(x_0) \).

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