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ON SPACES WHICH ARE NOT OF COUNTABLE CHARACTER

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It is well known that the unit interval I has a countable base and the fixed point property. By considering the maps $g(x)=x^2$ and $h(x)=1-x$, one sees that there is no $x\in I$ such that for every continuous map $f\colon I\rightarrow I$, $x\in f(I)$ implies $f(x)=x$.

In Theorem 1, it is shown that if A is a closed, non-null proper subset of a locally connected, compact Hausdorff space X which has a countable base, then there exists a continuous map $f\colon X\rightarrow X$ such that $A\cap f(X)$ is not contained in $A\cap f(A)$. Theorem 2 shows that certain nondegenerate topological spaces X contain proper subsets M such that for every continuous map $f\colon X\rightarrow X$, $M\cap f(X)\subset M\cap f(M)$. That is, for each of these spaces X and every continuous map $f\colon X\rightarrow X$, $x\in M\cap f(X)$ implies $f^{-1}(x)\cap M\neq\emptyset$. The corollary is of interest in that, if X satisfies the hypotheses of Theorem 2 and M consists of a single point, then a fixed point of some of the maps $f\colon X\rightarrow X$ is located.

THEOREM 1. *Suppose X is a connected, locally connected, compact Hausdorff space which has a countable base. If A is any non-null, closed, proper subset of X, then there exists a continuous map $f\colon X\rightarrow X$ such that $A\cap f(X)\backslash A\cap f(A)\neq\emptyset$.*

PROOF. Since X is compact Hausdorff and has a countable base, X is metrizable. Hence X is arcwise connected. Let $y\in X\backslash A$. Since

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X is normal, there exists a continuous map h such that $h(x)=0$ for $x\in A$, $h(y)=1$, and $0\leqq h(x)\leqq 1$ for each $x\in X$. Since X is arcwise connected, there is an arc C connecting y and A. Now C contains a subarc C_1, such that $y\in C_1$ and $C_1\cap A$ is a single point x_0. Then there is a homeomorphism g such that $g([0, 1])=C_1$, $g(0)=y$, and $g(1)=x_0$. Consider the continuous map $f=gh$. Clearly $f\colon X\to X$ and $x_0\in A\cap f(X)$. But since $x_0\neq y$ and $f(A)=y$, $x_0\notin f(A)$. Hence $x_0\notin A\cap f(A)$; and $f=gh$ is the required map.

In the following let M consist of the set of all points $x\in X$ such that if x is a limit point of $\{y_\alpha\}$ where $\bigcup y_\alpha\subset X\backslash x$, then $\{y_\alpha\}$ contains uncountably many distinct points. It may be noted that X does not satisfy the first axiom of countability at points of M.

THEOREM 2. *Let X be a connected Hausdorff space which contains a non-null set M such that $M=\overline{M}$, and $M\neq X$. Suppose also that each point of $X\backslash M$ has a countable base. Then for every continuous map $f\colon X\to X$, $M\cap f(X)\subset M\cap f(M)$.*

PROOF. Let f be a continuous function such that f maps X into X. If $M\cap f(X)=\emptyset$, then $M\cap f(X)\subset M\cap f(M)$. On the other hand, suppose $x\in M\cap f(X)$ and $x\notin f(M)$. Now x is a limit point of $X\backslash x$, for otherwise X would not be connected. Since $x\notin f(M)$, $f^{-1}(x)\subset X\backslash M$. Suppose there exists $z\in f^{-1}(x)$ such that every neighborhood of z intersects $X\backslash f^{-1}(x)$. Since $X\backslash M$ is open and $z\in X\backslash M$, there exists a countable set $\{U_n(z)\}$ of neighborhoods of z such that $\bigcap_{n=1}^{\infty} U_n(z)=z$, and $U_n(z)\subset X\backslash M$ for each n. In each $U_n(z)$ there exists a point u_n such that $u_n\in X\backslash f^{-1}(x)$. Now $f(u_n)\subset X\backslash x$ for each n; and, by the continuity of f, x is a limit point of the set $\bigcup_{n=1}^{\infty} f(u_n)$. But $\bigcup_{n=1}^{\infty} f(u_n)$ does not contain uncountably many distinct points. Thus a contradiction has been reached. Suppose that for every $z\in f^{-1}(x)$, there exists a neighborhood $U(z)$ such that $U(z)\cap\{X\backslash f^{-1}(x)\}=\emptyset$. Clearly $U(z)$ may be taken so that $U(z)\subset X\backslash M$. Then $U(z)\subset f^{-1}(x)$ for every $z\in f^{-1}(x)$, and $f^{-1}(x)$ is open in X. Since x is closed in X, $f^{-1}(x)$ is closed in X. Therefore $f^{-1}(x)=X$ and $f(X)=x$. But $f(M)\subset f(X)$; hence, $f(M)=x$. But this contradicts the assumption that $x\notin f(M)$.

COROLLARY. *If X is a nondegenerate connected Hausdorff space in which M is a single point x_0, then for every continuous function f such that $f\colon X\to X$ and $x_0\in f(X)$, $f(x_0)=x_0$.*

PROOF. By Theorem 2, $x_0\subset f(x_0)$. Since $f(x_0)$ is a single point, $x_0=f(x_0)$.

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