TOTALLY NONCONNECTED IM KLEINEN CONTINUA

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If a connected topological space \( T' \) is not connected im kleinen at the point \( p \) of \( T' \), then there is some open set \( D' \) containing \( p \) such that \( p \) is a boundary point of the \( p \)-component of \( D' \).

This paper shows that much stronger conditions than those above hold at some points of a Baire topological continuum\(^1\) which is not connected im kleinen at any point of a certain domain intersection subset. Limits to results of this kind are shown by an example of a bounded plane continuum that is connected im kleinen at each point of a dense inner limiting (i.e., \( G_\delta \)) subset but is not locally connected at any point.

**Definition.** A topological continuum \( T' \) is totally nonconnected im kleinen on a subset \( A \) of \( T' \) if \( T' \) is not connected im kleinen at any point of \( A \) (i.e., if each point \( p \) of \( A \) is contained in some open subset \( U \) of \( T' \) such that \( p \) is a boundary point, relative to \( T' \), of the \( p \)-component of \( U \)).

**Definition.** If (1) \( T' \) is a topological continuum, (2) \( Z' \) is the least cardinal number of any topological basis for \( T' \), (3) the set \( U \) is an open subset of \( T' \), and (4) the subset \( A \) of \( U \) is the nonvacuous common part of not more than \( Z' \) open subsets of \( T' \), each dense in \( U \), then \( A \) is a dense-domain intersection subset of \( U \) (relative to \( T' \)). If, in addition to (1), (2) and (3), \( U \) is not the sum of \( Z' \), or fewer than \( Z' \), closed nowhere dense subsets of \( T' \), then \( T' \) is Baire topological on \( U \). If \( T' \) is Baire topological on each open subset of \( U \), then \( T' \) is locally Baire topological on \( U \).

**Theorem 1.** If the topological continuum \( T' \) is Baire topological on the open subset \( U \), then \( U \) contains an open subset \( V \) such that \( T' \) is locally Baire topological on \( V \).

**Proof.** Let \( R'_1, R'_2, \ldots, R'_i, \ldots \) be a most economical well-ordering of a basis for \( T' \) of minimum cardinality. Assume \( T' \) is not locally Baire topological on any open subset of \( U \). Then there is a subsequence \( R'_{N(1)}, R'_{N(2)}, \ldots, R'_{N(i)}, \ldots \) of \( R'_1, R'_2, \ldots, R'_i, \ldots \) such that, for each \( i \), there is a dense-domain intersection subset \( I_i \) of \( T' \) such that \( I_i \cdot R'_{N(i)} = 0 \) and each open subset of \( U \) contains some term of \( R'_{N(1)}, R'_{N(2)}, \ldots, R'_{N(i)}, \ldots \). But \( \prod_i I_i \) is the common part...

\(^1\) The terminology of this paper is that of [1].
of not more than \( Z' \) dense-domain intersection subsets of \( T \), where \( Z' \) is the cardinality of \( R'_1, R'_2, \ldots, R'_i, \ldots \). Consequently [1, Theorem 3], the set \( \prod_i I_i \) is a dense-domain intersection subset of \( T \). But [1, Theorem 1], the closure of any dense-domain intersection subset of \( T \) contains an open subset of \( U \), and therefore, the closure of \( \prod_i I_i \) contains an open subset of \( U \). This is a contradiction.

**Standing notation.** \( T \) is a topological continuum which is locally Baire topological on the open set \( D \) and is totally nonconnected im kleinen on a dense-domain intersection subset \( I \) of \( D \). The least cardinal number of a basis for \( T \) is \( Z \). The collection \( R \) of regions is a basis for \( T \) of cardinality \( Z \). The sequence \( R_1, R_2, \ldots, R_i, \ldots \), is a most economical well-ordering of \( R \).

**Lemma 1.** Let \( U \) be an open subset of \( D \) and let \( S \) be the set of all points \( p \) such that \( p \) is in \( U - U \) or \( p \) is in \( U \) and \( p \) is a boundary point of the \( p \)-component of \( U \). Then \( S \) is closed.

**Proof.** Assume there is a point \( q \) in \( S \) which is not in \( S \). Then \( q \) is in \( U \) since \( U \supset S \) and each point of \( U - U \) is in \( S \). Therefore, each region containing \( q \) contains a point \( r \) of \( S \cdot U \) and hence contains a point of \( U \) not in the \( q \)-component of \( U \). Consequently, \( q \) is in \( S \). This is a contradiction and so \( S \) is closed.

**Lemma 2.** Let \( V \) be an open subset of \( D \). Then \( V \) contains an open set no component of which contains an open set.

**Proof.** Let \( R_{M(1)}, R_{M(2)}, \ldots, R_{M(i)}, \ldots \) be the subsequence of \( R_1, R_2, \ldots, R_i, \ldots \) consisting of those terms contained in \( V \). For each \( i \), let \( S_i \) be the set related to \( R_{M(i)} \) as \( S \) is related to \( U \) in Lemma 1. The continuum \( T \) is totally nonconnected im kleinen on the dense-domain intersection subset \( I \cdot V \) of \( V \) and therefore \( \sum_i S_i \supset I \cdot V \). But, since \( I \) is a dense-domain intersection subset of \( D \), the set \( D - I \) is contained in the sum of not more than \( Z \) closed sets, each nowhere dense in \( D \). Hence, if each term of \( S_1, S_2, \ldots \), were nowhere dense, then \( V \) would be contained in the sum of not more than \( Z \) closed, nowhere dense subsets of \( T \). But that can not be, since \( T \) is Baire topological on \( V \). Consequently, for some \( i \), the set \( S_i \) contains an open set \( V' \). By definition of \( S \), no component of the open set \( V \cdot V' \) can contain an open set.

**Theorem 2.** The open set \( D \) contains an open set \( D' \), dense in \( D \), no component of which contains an open set.

**Proof.** Let \( D_1 \) be an open subset of \( D \), no component of which contains an open set. If \( i \) is an ordinal greater than 1 such that the
closure of $\sum_{j<i} D_j$ does not contain $D$, then let $D_i$ be an open subset of the complement, in $D$, of the closure of $\sum_{j<i} D_j$, no component of which contains an open set. Let $D' = \sum_i D_i$. Then $D'$ is an open subset of $D$, dense in $D$, no component of which contains an open set.

**Theorem 3.** Let the topological continuum $T'$ be locally Baire topological on the open set $V$. Then $T'$ is totally nonconnected im kleinen on a dense-domain intersection subset of $V$ if and only if there is an open subset of $V$, dense in $V$, no component of which contains an open set.

**Theorem 4.** If $T$ is regular then $T$ contains a dense-domain intersection subset $J$ of $D$, contained in $I$, such that if $p$ is a point of $J$ and $T$ is nonaposyndetic at $p$ with respect to $T - \overline{R}_i$, where $R_i$ contains $p$, then the $p$-component of $\overline{R}_i$ does not contain an open set.

**Proof.** The collection $G$ of complements of closures of regions of $R$ has the Z-domain property and $T$ is totally nonaposyndetic on $I$ with respect to $G$. Let $P$ be a dense subset of $T$ of cardinality not greater than $Z$. The author has proved [1, Theorem 8] that under these circumstances there is a dense-domain intersection subset $J$ of $D$, contained in $I$, such that if $q$ is a point in $J$ and $T$ is nonaposyndetic at $q$ with respect to $T - \overline{R}_i$ of $G$ then $T - \overline{R}_i$ cuts (weakly) $q$ from each point of $P \cdot \overline{R}_i$. In which case, the $p$-component of $\overline{R}_i$ does not contain an open set.

The following example of a plane continuum, each connected open subset of which is dense in it, shows that Theorem 3 does not hold if “totally nonconnected im kleinen on a dense-domain intersection subset of $V$” is replaced by “not locally connected at any point of $V$”; even in the case where $V = T'$.

Let $H_1, H_2, \cdots$ be the sequence of closed plane point sets defined by induction as follows. Let $S$ denote the unit disk and let $H_1$ be the logical sum of the closed (topological) disk sequence $J_1, J_2, \cdots$ indicated in Figure 1. The sum of the straight portions of $J_1, J_2, \cdots, J_4$ is the limiting set of $J_1, J_2, \cdots$.

Assume $H_i$ to be defined and let $J'_1, J'_2, \cdots$ be a counting of the closed disks maximal in $H_i$. For each natural number $j$, there exist two points $r$ and $s$ of $J'_j$ such that $r+s$ cuts $J'_j$ from $H_i - J'_j$ in $H_i$ and such that if a single point $t$ does the same cutting then $t = r$. Let $f_j$ be a homeomorphism of $S$ onto $J'_j$ such that $f_j(p) = r$ and $f_j(q) = s$. Let $H_{i+1} = \sum_j f_j(H_i)$. The homeomorphisms of $S$ onto the closed maximal disks of the sets $H_2, H_3, \cdots$ are chosen in such a way that the diameters of disks in $H_1, H_2, \cdots$ are eventually as small as one wishes.
Let $H = \prod_i H_i$. It is clear that $H$ is not locally connected at any point but $H$ is connected im kleinen at each point of the dense inner limiting subset of $H$ consisting of the points that, for each $i$, are interior points (in the plane) of $H_i$. Hence $H$ does not contain any open set no component of which contains an open set.

**Bibliography**