TOTALLY NONCONNECTED IM KLEINEN CONTINUA

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If a connected topological space $T'$ is not connected im kleinen at the point $p$ of $T'$, then there is some open set $D'$ containing $p$ such that $p$ is a boundary point of the $p$-component of $D'$.

This paper shows that much stronger conditions than those above hold at some points of a Baire topological continuum which is not connected im kleinen at any point of a certain domain intersection subset. Limits to results of this kind are shown by an example of a bounded plane continuum that is connected im kleinen at each point of a dense inner limiting (i.e., $G_4$) subset but is not locally connected at any point.

**Definition.** A topological continuum $T'$ is **totally nonconnected im kleinen** on a subset $A$ of $T'$ if $T'$ is not connected im kleinen at any point of $A$ (i.e., if each point $p$ of $A$ is contained in some open subset $U$ of $T'$ such that $p$ is a boundary point, relative to $T'$, of the $p$-component of $U$).

**Definition.** If (1) $T'$ is a topological continuum, (2) $Z'$ is the least cardinal number of any topological basis for $T'$, (3) the set $U$ is an open subset of $T'$, and (4) the subset $A$ of $U$ is the nonvacuous common part of not more than $Z'$ open subsets of $T'$, each dense in $U$, then $A$ is a **dense-domain intersection subset** of $U$ (relative to $T'$). If, in addition to (1), (2) and (3), $U$ is not the sum of $Z'$, or fewer than $Z'$, closed nowhere dense subsets of $T'$, then $T'$ is **Baire topological** on $U$. If $T'$ is Baire topological on each open subset of $U$, then $T'$ is **locally Baire topological** on $U$.

**Theorem 1.** If the topological continuum $T'$ is Baire topological on the open subset $U$, then $U$ contains an open subset $V$ such that $T'$ is locally Baire topological on $V$.

**Proof.** Let $R'_1, R'_2, \ldots, R'_i, \ldots$ be a most economical well-ordering of a basis for $T'$ of minimum cardinality. Assume $T'$ is not locally Baire topological on any open subset of $U$. Then there is a subsequence $R'_{N(1)}, R'_{N(2)}, \ldots, R'_{N(i)}, \ldots$ of $R'_1, R'_2, \ldots, R'_i, \ldots$ such that, for each $i$, there is a dense-domain intersection subset $I_i$ of $T'$ such that $I_i \cap R'_{N(i)} = \emptyset$ and each open subset of $U$ contains some term of $R'_{N(1)}, R'_{N(2)}, \ldots, R'_{N(i)}, \ldots$. But $\prod_i I_i$ is the common part

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1 The terminology of this paper is that of [1].
of not more than $Z'$ dense-domain intersection subsets of $T$, where
$Z'$ is the cardinality of $R_1', R_2', \ldots, R_i', \ldots$. Consequently
[1, Theorem 3], the set $\prod_i I_i$ is a dense-domain intersection subset
of $T$. But [1, Theorem 1], the closure of any dense-domain inter-
section subset of $T$ contains an open subset of $U$, and therefore, the
closure of $\prod_i I_i$ contains an open subset of $U$. This is a contradiction.

**Standing notation.** $T$ is a topological continuum which is locally
Baire topological on the open set $D$ and is totally nonconnected im
kleinen on a dense-domain intersection subset $I$ of $D$. The least
Cardinal number of a basis for $T$ is $Z$. The collection $R$ of regions is a
basis for $T$ of cardinality $Z$. The sequence $R_1, R_2, \ldots, R_i, \ldots,$
is a most economical well-ordering of $R$.

**Lemma 1.** Let $U$ be an open subset of $D$ and let $S$ be the set of all
points $p$ such that $p$ is in $U - U$ or $p$ is in $U$ and $p$ is a boundary point
of the $p$-component of $U$. Then $S$ is closed.

**Proof.** Assume there is a point $q$ in $S$ which is not in $S$. Then $q$
is in $U$ since $U \supset S$ and each point of $U - U$ is in $S$. Therefore, each
region containing $q$ contains a point $r$ of $S \cup U$ and hence contains a
point of $U$ not in the $p$-component of $U$. Consequently, $q$ is in $S$. This
is a contradiction and so $S$ is closed.

**Lemma 2.** Let $V$ be an open subset of $D$. Then $V$ contains an open
set no component of which contains an open set.

**Proof.** Let $R_{M(1)}, R_{M(2)}, \ldots, R_{M(i)}, \ldots$ be the subsequence of
$R_1, R_2, \ldots, R_i, \ldots$ consisting of those terms contained in $V$. For
each $i$, let $S_i$ be the set related to $R_{M(i)}$ as $S$ is related to $U$ in Lemma
1. The continuum $T$ is totally nonconnected im kleinen on the dense-
domain intersection subset $I \cdot V$ of $V$ and therefore $\sum_i S_i \supset I \cdot V$. But,
since $I$ is a dense-domain intersection subset of $D$, the set $D - I$ is
contained in the sum of not more than $Z$ closed sets, each nowhere
dense in $D$. Hence, if each term of $S_1, S_2, \ldots$ were nowhere dense,
then $V$ would be contained in the sum of not more than $Z$ closed,
nowhere dense subsets of $T$. But that can not be, since $T$ is Baire
topological on $V$. Consequently, for some $i$, the set $S_i$ contains an
open set $V'$. By definition of $S_i$, no component of the open set $V \cdot V'$
can contain an open set.

**Theorem 2.** The open set $D$ contains an open set $D'$, dense in $D$, no
component of which contains an open set.

**Proof.** Let $D_1$ be an open subset of $D$, no component of which
contains an open set. If $i$ is an ordinal greater than 1 such that the
closure of $\sum_{j \leq i} D_j$ does not contain $D$, then let $D_i$ be an open subset of the complement, in $D$, of the closure of $\sum_{j \leq i} D_j$, no component of which contains an open set. Let $D' = \sum_i D_i$. Then $D'$ is an open subset of $D$, dense in $D$, no component of which contains an open set.

**Theorem 3.** Let the topological continuum $T'$ be locally Baire topological on the open set $V$. Then $T'$ is totally nonconnected im kleinen on a dense-domain intersection subset of $V$ if and only if there is an open subset of $V$, dense in $V$, no component of which contains an open set.

**Theorem 4.** If $T$ is regular then $T$ contains a dense-domain intersection subset $J$ of $D$, contained in $I$, such that if $p$ is a point of $J$ and $T$ is nonaposyndetic at $p$ with respect to $T - \overline{R_i}$, where $R_i$ contains $p$, then the $p$-component of $\overline{R_i}$ does not contain an open set.

**Proof.** The collection $G$ of complements of closures of regions of $R$ has the Z-domain property and $T$ is totally nonaposyndetic on $I$ with respect to $G$. Let $P$ be a dense subset of $T$ of cardinality not greater than $Z$. The author has proved [1, Theorem 8] that under these circumstances there is a dense-domain intersection subset $J$ of $D$, contained in $I$, such that if $q$ is a point in $J$ and $T$ is nonaposyndetic at $q$ with respect to $T - \overline{R_i}$ of $G$ then $T - \overline{R_i}$ cuts (weakly) $q$ from each point of $P \cdot \overline{R_i}$. In which case, the $p$-component of $\overline{R_i}$ does not contain an open set.

The following example of a plane continuum, each connected open subset of which is dense in it, shows that Theorem 3 does not hold if “totally nonconnected im kleinen on a dense-domain intersection subset of $V$” is replaced by “not locally connected at any point of $V”$; even in the case where $V = T'$.

Let $H_1, H_2, \cdots$ be the sequence of closed plane point sets defined by induction as follows. Let $S$ denote the unit disk and let $H_1$ be the logical sum of the closed (topological) disk sequence $J_1, J_2, \cdots$ indicated in Figure 1. The sum of the straight portions of $J_1, J_2, \cdots$ is the limiting set of $J_1, J_2, \cdots$.

Assume $H_i$ to be defined and let $J'_1, J'_2, \cdots$ be a counting of the closed disks maximal in $H_i$. For each natural number $j$, there exist two points $r$ and $s$ of $J'_j$ such that $r+s$ cuts $J'_j$ from $H_i - J'_j$ in $H_i$ and such that if a single point $t$ does the same cutting then $t = r$. Let $f_j$ be a homeomorphism of $S$ onto $J'_j$ such that $f_j(p) = r$ and $f_j(q) = s$. Let $H_{i+1} = \sum_j f_j(H_i)$. The homeomorphisms of $S$ onto the closed maximal disks of the sets $H_2, H_3, \cdots$ are chosen in such a way that the diameters of disks in $H_1, H_2, \cdots$ are eventually as small as one wishes.
Let $H = \prod_i H_i$. It is clear that $H$ is not locally connected at any point but $H$ is connected im kleinen at each point of the dense inner limiting subset of $H$ consisting of the points that, for each $i$, are interior points (in the plane) of $H_i$. Hence $H$ does not contain any open set no component of which contains an open set.

**Bibliography**


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