THE COHOMOLOGY ALGEBRA OF CERTAIN LOOP SPACES

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The purpose of this paper[1] is to determine the cohomology algebra of a loop space over a topological space whose cohomology algebra is a truncated polynomial algebra generated by an element of even degree. As special cases we obtain the well-known results when the base space has as cohomology algebra an exterior algebra (the base space an even dimensional sphere) or a polynomial algebra (the base space infinite dimensional complex projective space; compare also Theorem 2 in [**1**]). In particular, the result is applicable to loop spaces over complex and quaternionic projective n-spaces and the Cayley plane.

Throughout, A will denote a commutative ring with unit and *A-algebra* will mean an associative A-algebra with unit.

1. Augmented spectral sequences of algebras.

A *differential A-module* consists of an A-module E and a (module) endomorphism $d: E \to E$ such that $dd=0$. The map d is called a *differential* and the elements of its kernel and image are called *cycles* and *boundaries* respectively; the quotient module $H(E)=$Kernel of d/Image of d is called the *derived module.* A *differential A-algebra* consists of an A-algebra which is a differential A-module and an automorphism $\omega: E \to E$ such that

$$(1.1) \qquad d\omega + \omega d = 0, \qquad d(xy) = (dx)y + \omega(x)dy, \qquad x, y \in E.$$

It follows that $H(E)$ has a naturally induced multiplication under which $H(E)$ is an A-algebra. An *augmentation* of a differential A-algebra is an algebra homomorphism $\alpha: E \to A$ with right inverse $\beta: A \to E$ such that $\alpha d=0$. It follows that $H(E)$ has a naturally induced augmentation $\bar{\alpha}$. The kernel of α will be denoted by E^+.

An *augmented spectral sequence of A-algebras* is a sequence of augmented differential A-algebras (E_r), $r \geqq 0$, such that $E_{r+1}=H(E_r)$ and $\alpha_{r+1}=\bar{\alpha}_r$. The *limit* of (E_r) is the augmented A-algebra defined as follows: An element $x_r \in E_r$ is called a *permanent cycle* if it is a cycle and its successive projections in $E_{r+1}, E_{r+2}, \cdots$ are cycles. Let E_∞ be the set of sequences (x_r) where x_r is a permanent cycle of E_r and x_{r+1} is the projection of x_r in E_{r+1}, with two such sequences identified if $x_r = x_r'$ for all $r \geqq r_0$. Defining

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$$(1.2)\qquad \begin{aligned} (x_r)+(y_r) &= (x_r+y_r), & a(x_r) &= (ax_r), & a\in A,\\ (x_r)(y_r) &= (x_ry_r), & \alpha(x_r) &= \alpha_r x_r, \end{aligned}$$

where α_r is the augmentation of E_r, makes E_∞ into an augmented A-algebra. The augmentation α of E_∞ is well-defined and its kernel E_∞^+ is the subalgebra defined by permanent cycles (x_r) such that $x_r \in E_r^+$. The spectral sequence is *acyclic* if $E_\infty^+=0$.

An augmented spectral sequence of A-algebras is *canonical* if the sequence (E_r) is defined for $r\geqq 2$ and for each r:

(a) E_r is a bigraded algebra, $E_r=\sum_{p,q} E_r^{p,q}$, with $E_r^{p,q}=0$ if $p<0$ or $q<0$; moreover, the multiplication in E_r is anticommutative with respect to total degree $p+q$.

(b) The differential d_r is bihomogeneous of bidegree $(r, 1-r)$.

(c) The automorphism ω_r is given by $\omega_r(x)=(-1)^{p+q}x$ for $x\in E_r^{p,q}$.

(d) The augmentation α_r maps $E_r^{0,0}$ isomorphically onto A.

(e) $E_{r+1}^{p,q}=H(E_r^{p,q})$.

It follows from (d) that $E_r^+=\sum_{p+q>0} E_r^{p,q}$. From (e) it follows that E_∞ has a naturally induced bigrading with $E_\infty^{p,q}=0$ if $p<0$ or $q<0$. In view of (d) it then follows that α maps $E_\infty^{0,0}$ isomorphically onto A, and $E_\infty^+=\sum_{p+q>0} E_\infty^{p,q}$. Thus acyclicity of the spectral sequence is equivalent to the statement that $E_\infty^{p,q}=0$ for $p+q>0$. It may be readily proved that

$$(1.3)\qquad E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q} \qquad \text{if } r>p \text{ and } r>q+1.$$

The spectral sequence is said to be *initially decomposable* if

$$(1.4)\qquad E_2^{p,q} = E_2^{p,0}\cdot E_2^{0,q};$$

more precisely, if every $y\in E_2^{p,q}$ can be written as a sum of products xz where $x\in E_2^{p,0}$ and $z\in E_2^{0,q}$. Note that $B=\sum E_2^{p,0}$ and $F=\sum E_2^{0,q}$ are graded subalgebras of E_2.

2. **Monogenic twisted polynomial algebras.** A *monogenic twisted polynomial A-algebra of height h, $2\leqq h\leqq\infty$, and type $t=(t_{m,n})$* is a free A-module generated by a sequence of elements $x_0, x_1, \cdots, x_{h-1}$ with multiplication defined by

$$(2.1)\qquad x_mx_n = \begin{cases} t_{m,n}x_{m+n} & m+n<h,\\ 0 & m+n\geqq h, \end{cases}$$

where the $t_{m,n}$ are nonzero elements of A which satisfy:

$$(2.2)\qquad t_{0,n}=1, \qquad t_{m,0}=1,$$

$$(2.3)\qquad t_{m,n}=t_{n,m},$$

$$(2.4) \qquad t_{m,n}t_{m+n,k} = t_{m,n+k}t_{n,k}.$$

From (2.3) and (2.4) follow commutativity and associativity respectively; from (2.1) and (2.2) follows that $x_0=1$. The powers x_1^m are related to the generators x_m as follows: Putting $t_k=t_{1,k-1}$, $(k>0)$, then by induction one proves

$$(2.5) \qquad x_1^m = t_1t_2 \cdots t_m x_m.$$

We shall write $x_1=x$ and denote the algebra by $A[x, h, t]$. In particular, if $t_{m,n}=1$ for all $m+n<h$ then the algebra is the ordinary (truncated) polynomial algebra of height h which we shall denote by $A[x, h]$; evidently $A[x, 2]$ is the exterior algebra $\Lambda_A(x)$. If each $t_{m,n}$ differs from the binomial coefficient $(m, n)=m+n!/m!n!$ by a unit then the algebra will be said to be of *binomial type.*

A *monogenic twisted A-algebra of binomial type* is a free A-module generated by a sequence of elements $(x_0, x_1, x_2, \cdots)$ with multiplication defined by

$$(2.6) \qquad x_m x_n = (m, n)x_{m+n}.$$

It will be denoted by $T_A(x_0, x_1, x_2, \cdots)$. Since the binomial coefficients satisfy (2.2), (2.3), and (2.4), $T_A(x_0, x_1, x_2, \cdots)$ is associative, commutative, and $x_0=1$.

We note the following readily proved property of the binomial coefficients (m, n) modulo a prime p:

$$(2.7) \qquad (m, n) = (m_0, n_0)(m_1, n_1) \cdots (m_j, n_j),$$

where

$$(2.8) \qquad m = m_0 + m_1p + \cdots + m_ip^i, \quad n = n_0 + n_1p + \cdots + n_jp^j, \quad i \leqq j,$$

are the p-adic expansions of m and n, and $m_k=0$ if $k>i$.

PROPOSITION 1. (a) *If A has characteristic zero then*

$$(2.9) \qquad T_A(x_0, x_1, x_2, \cdots) = A[x, \infty, t].$$

(b) *If A has characteristic prime p then there is an algebra isomorphism*[2]

$$(2.10) \qquad \phi: T_A(x_0, x_1, x_2, \cdots) \cong \otimes_{i\geqq 0} A[x_{p^i}, p, t^{(i)}], \quad t_{m,n}^{(i)} = (m, n).$$

PROOF. To prove (a) we need only note that $(m, n)\neq 0$. (b) For each $i\geqq 0$ define p elements $y_m^{(i)}=x_{mp^i}$, $0\leqq m<p$. If $0\leqq m<p$ and $0\leqq n<p$ then, using (2.6) and (2.7), we have

[2] By $\otimes_{i\geqq 0}$ is meant the "weak" tensor product.

$$y_m^{(i)} y_n^{(i)} = x_{mp^i} x_{np^i} = (mp^i, np^i) x_{(m+n)p^i} = (m, n) x_{(m+n)p^i}.$$

It is clear that $(m, n) \equiv 0$ if and only if $m+n \geqq p$; hence

$$y_m^{(i)} y_n^{(i)} = \begin{cases} (m, n) y_{m+n}^{(i)} & m+n < p, \\ 0 & m+n \geqq p. \end{cases}$$

Thus (2.1) is satisfied; the remaining conditions (2.3), (2.4), and (2.5) are also satisfied as noted previously. Thus for each i we have a subalgebra $A[y^{(i)}, p, t^{(i)}]$ (of binomial type). It remains to show that $T_A(x_0, x_1, x_2, \cdots)$ is isomorphic to their tensor product.

It follows from (2.6) and (2.7) that corresponding to the p-adic expansion (2.8) for m, we may write x_m as a unique product

$$x_m = x_{m_0} x_{m_1 p} \cdots x_{m_i p^i} = y_{m_0}^{(0)} y_{m_1}^{(1)} \cdots y_{m_i}^{(i)}.$$

Therefore the correspondence defined by

$$\phi(x_m) = y_{m_0}^{(0)} \otimes y_{m_1}^{(1)} \otimes \cdots \otimes y_{m_i}^{(i)}$$

establishes a module isomorphism (2.10). It remains to show that ϕ is multiplicative. Let (2.8) be the p-adic expansions of m and n (we may assume $i \leqq j$). Then, using (2.6) and (2.7),

$$\begin{aligned} \phi(x_m)\phi(x_n) &= (y_{m_0}^{(0)} \otimes \cdots \otimes y_{m_i}^{(i)})(y_{n_0}^{(0)} \otimes \cdots \otimes y_{n_j}^{(j)}) \\ &= (m_0, n_0) \cdots (m_j, n_j) y_{m_0+n_0}^{(0)} \otimes \cdots \otimes y_{m_j+nj}^{(j)} \\ &= (m, n) y_{m_0+n_0}^{(0)} \otimes \cdots \otimes y_{m_j+n_j}^{(j)}. \end{aligned} \tag{2.11}$$

We consider 2 cases:

(i) If $(m, n) \equiv 0 \mod p$ then $\phi(x_m)\phi(x_n) = 0$. But also $\phi(x_m x_n) = \phi(0) = 0$.

(ii) If $(m, n) \not\equiv 0 \mod p$ then none of the factors (m_r, n_r), $0 \leqq r \leqq j$, are zero and hence $m_r + n_r < p$ for all r. But then

$$m + n = (m_0 + n_0) + (m_1 + n_1)p + \cdots + (m_j + n_j)p^j,$$

$$\phi(x_{m+n}) = y_{m_0+n_0}^{(0)} \otimes \cdots \otimes y_{m_j+n_j}^{(j)}.$$

Thus (2.11) becomes

$$\phi(x_m)\phi(x_n) = (m, n)\phi = \phi(x_{m+n}),$$

completing the proof of Proposition 1.

3. The main theorem.

THEOREM 1. *Let (E_r) be an initially decomposable acyclic canonical spectral sequence of A-algebras. If B is a truncated polynomial algebra*

$A[x, h]$ where x has even degree $m \geqq 2$, then

$$F \cong \Lambda_A(x_1) \otimes T_A(z_0, z_2, z_4, \cdots) \tag{3.1}$$

where $z_1 \in F$ and is of degree $m-1$ and $z_{2i} \in F$ and has degree $i(hm-2)$.

In view of Proposition 1 we have:

COROLLARY. (a) *If in addition A has characteristic zero then*

$$F \cong \Lambda_A(z_1) \otimes A[z_2, \infty, t], \tag{3.2}$$

the second factor being of binomial type.

(b) *If in addition A has prime characteristic p then*

$$F \cong \Lambda_A(z_1) \otimes_{i \geqq 0} A[z_{2p^i}, p, t^{(i)}], \tag{3.3}$$

each of the monogenic factors in the tensor product being of binomial type.

PROOF. The following is a trivial consequence of (1.5):

$$E_r^{p,q} = 0 \quad \text{if } E_2^{p,0} = 0 \text{ or } E_2^{0,q} = 0 \qquad (r \geqq 2). \tag{3.4}$$

We note further that the assumption on B gives

$$E_2^{p,0} = 0, \qquad \text{if } p \neq tm, \ (t = 0, 1, \cdots, h-1), \tag{3.5}$$

$$E_2^{tm,0} = \begin{cases} 0 & \text{if } t \geqq h, \\ A \cdot x^t & \text{if } 0 \leqq t < h, \end{cases} \tag{3.6}$$

(by $A \cdot x^t$ we mean the free A-module generated by x^t). We shall now prove:

$$E_2 = E_3 = \cdots = E_m, \tag{3.7}$$

$$E_{tm+1} = E_{tm+2} = \cdots = E_{(t+1)m}, \qquad t \geqq 1, \tag{3.7$'$}$$

$$E_{(h-1)m+1} = E_{(h-1)m+2} = \cdots = E_\infty. \tag{3.7$''$}$$

If $p \neq sm$ then, in view of (3.4) and (3.6), $E_r^{p,q} = 0$ and hence $d_r(E_r^{p,q}) = 0$. On the other hand, if $p = sm$ but r is not a multiple of m then

$$d_r(E_r^{sm,q}) \subset E_r^{sm+r,q-r+1} = 0$$

(the latter module is zero by (3.4) and (3.6) since $sm+r$ is not a multiple of m). Thus $d_r = 0$ if r is not a multiple of m and (3.7) and (3.7)$'$ follow. If $r \geqq hm$ then by (3.4) and (3.6), $E_r^{p+r,q-r+1} = 0$ and hence $d_r(E_r^{p,q}) = 0$. Thus $E_{hm} = E_{hm+1} = \cdots = E_\infty$. Combining this with (3.7)$'$ (taking $t = h-1$) gives (3.7)$''$.

REMARK. Using these results it will be possible to identify $E_2^{0,q} = E_r^{0,q}$ for some values of q and r $(r>2)$. When we write $d_r(u)$, where $u \in E_2^{0,q}$, such an identification will be implied.

For convenience put $q_0 = hm-2$. We shall now prove:

A. If $q \not\equiv 0,\ m-1$, modulo q_0 then $E_2^{0,q}=0$.

B. There exists a sequence of generators $z_0=1,\ z_1,\ z_2,\ \cdots,\ z_j,\ \cdots$ for F such that

$$(3.8)\qquad E_2^{0,jq_0+(m-1)} = A\cdot z_{2j+1}, \qquad d_m(z_{2j+1}) = xz_{2j},$$

$$(3.9)\qquad E_2^{0,jq_0} = A\cdot z_{2j}, \qquad d_{(h-1)m}(z_{2j}) = x^{h-1}z_{2j-1},$$

(note the preceding remark).

The proof is by induction on q. Let $\bar{q}>0$ and assume:

$\mathrm{A}_{\bar{q}}$. Statement A holds for all q such that $0 \leqq q < \bar{q}$.

$\mathrm{B}_{\bar{q}}$. We have chosen generators $z_0=1$, z_{2j} $(0<jq_0<\bar{q})$, and z_{2j+1} $(m-1 \leqq jq_0+m-1<\bar{q})$ such that (3.8) and (3.9) hold.

Clearly A_1 and B_1 are trivial; it remains to prove $\mathrm{A}_{\bar{q}+1}$ and $\mathrm{B}_{\bar{q}+1}$. We shall first prove

$$(3.10)\qquad E_{tm+1}^{0,\bar{q}} = E_{tm}^{0,\bar{q}}$$

holds in the following cases:

(i) $\bar{q} \equiv 0 \pmod{q_0}$, $1 \leqq t < h-1$.

(ii) $\bar{q} \equiv m-1 \pmod{q_0}$, $1 < t \leqq h-1$.

(iii) $\bar{q} \not\equiv 0,\ m-1 \pmod{q_0}$, $1 \leqq t \leqq h-1$.

Consider

$$(3.11)\qquad 0 \xrightarrow{d_{tm}} E_{tm}^{0,\bar{q}} \xrightarrow{d_{tm}} E_{tm}^{tm,\bar{q}-tm+1}.$$

Since $d_{tm}d_{tm}=0$, to prove (3.10) it suffices to show that the last module in (3.11) is zero. If $\bar{q}-tm+1<0$ this is trivial; hence assume $\bar{q}-tm+1 \geqq 0$. We may write $\bar{q}=jq_0+s$, $0 \leqq s < q_0$.

(i) If $s=0$ and $1 \leqq t < h-1$ then $\bar{q}-tm+1 = jq_0-tm+1 \not\equiv 0$, $m-1 \pmod{q_0}$. Therefore it follows from $\mathrm{A}_{\bar{q}}$ and (3.4) that $E_{tm}^{tm,\bar{q}-tm+1}=0$.

(ii) $s=m-1$ and $1<t \leqq h-1$ then $\bar{q}-tm+1 = jq_0+(1-t)m \not\equiv 0$, $m-1 \pmod{q_0}$. As in the preceding case we may conclude that $E_{tm}^{tm,\bar{q}-tm+1}=0$.

(iii) Suppose $s \neq 0$, $m-1$ and $1 \leqq t \leqq h-1$. We have $\bar{q}-tm+1 \equiv s-tm+1 \pmod{q_0}$. If further $s \neq tm-1$, $(t+1)m-2$, then $\bar{q}-tm+1 \not\equiv 0$, $m-1 \pmod{q_0}$, and as in the preceding two cases we may conclude that the last module in (3.11) vanishes. It remains to consider the two exceptional cases:

If $s=tm-1$, then $\bar{q}-tm+1=jq_0$. Consider the map

$$(3.12)\qquad d_m\colon E_m^{(t-1)m,jq_0+m-1} \to E_{mq}^{tm,jq_0}.$$

Using (3.7) and (1.4) we have

$$(3.13)\qquad E_m^{tm,jq_0} = E_2^{tm,jq_0} = E_2^{tm,0}\cdot E_2^{0,jq_0}.$$

By hypothesis $E_2^{tm,0}=A\cdot x^t$, and by $B_{\bar q}$, $E_2^{0,jq_0}=A\cdot z_{2j}$. It follows from (3.13) that $E_m^{tm,jq_0}=A\cdot x^t z_{2j}$. Similarly we may show $E_m^{(t-1)m,jq_0+m-1} =A\cdot x^{t-1}z_{2j+1}$. Now using the latter part of (3.8) we have

$$d_m(x^{t-1}z_{2j+1}) = d_m(x^{t-1})z_{2j+1} + x^{t-1}d_m(z_{2j+1}) = x^t z_{2j}.$$

Thus (3.12) is an isomorphism and $E_{m+1}^{tm,jq_0}=0$. It follows that the last module in (3.11) is zero.

If $s=(t+1)m-2$, then $\bar q-tm+1=jq_0+m-1$. Since (3.12) is an isomorphism we have that $E_{m+1}^{(t-1)m,jq_0+m-1}=0$. The last module in (3.11) is therefore zero. This completes the proof of (3.10).

Proof of $A_{\bar q+1}$. Let $\bar q\not\equiv 0$, $m-1 \pmod{q_0}$. Using (3.7), (3.7)′, (3.7)″, and (3.10) (case iii), we may write $E_2^{0,\bar q}=E_\infty^{0,\bar q}$ which is zero by acyclicity.

Proof of $B_{\bar q+1}$. We assert that the following maps are isomorphisms:

$$(3.14)\qquad d_m\colon E_m^{0,\bar q}\to E_m^{m,\bar q-m+1},\qquad \text{if } \bar q = j_{q0}+m-1;$$

$$(3.15)\qquad d_{(h-1)m}\colon E_{(h-1)m}^{0,\bar q}\to E_{(h-1)m}^{(h-1)m,\bar q-(h-1)m+1}\qquad \text{if } \bar q = jq_0.$$

Assuming this we may prove B_{i+1} as follows: Let $\bar q=jq_0+m-1$; then $\bar q-m+1=jq_0<\bar q$. By (3.9), therefore $E_2^{0,jq_0}=A\cdot z_{2j}$. Since also $E_2^{m,0}=A\cdot x$, it follows from (1.4) that $E_2^{m,jq_0}=A\cdot xz_{2j}$. Using (3.7) we may replace the subscript m by 2 in each module in (3.14). If we therefore define $z_{2j+1}=d_m^{-1}(xz_{2j})$ then (3.8) holds for $B_{\bar q+1}$. Now let $\bar q=jq_0$. Then

$$\bar q-(h-1)m+1 = jq_0-(h-1)m+1 = (j-1)q_0+m-1.$$

Since also $\bar q-(h-1)m+1<\bar q$, we have by (3.8) $E_2^{0,\bar q-(h-1)m+1}=A\cdot z_{2j-1}$. By hypothesis, $E_2^{(h-1)m,0}=A\cdot x^{h-1}$; hence it follows from (1.4) that

$$(3.16)\qquad E_2^{(h-1)m,\bar q-(h-1)m+1} = A\cdot x^{h-1}z_{2j-1}.$$

We assert that we may identify

$$(3.17)\qquad E_{(h-1)m}^{(h-1)m,q-(h-1)m+1} = E_2^{(h-1)m,\bar q-(h-1)m+1}.$$

To prove this let $1\leqq t<h-1$ and consider

$$(3.18)\qquad E_{tm}^{(h-1-t)m,s}\xrightarrow{d_{tm}} E_{tm}^{(h-1)m,\bar q-(h-1)m+1}\xrightarrow{d_{tm}} E_{tm}^{(h-1+t)m,S}$$

where s and S are the appropriate integers. The last module is evidently zero (by 3.4) since $h-1+t\geqq h$. Also

$$s = \bar{q} - (h-1)m + 1 + tm - 1 = (j-1)q_0 + (t+1)m - 2.$$

Evidently $0<(t+1)m-2<q_0$ and hence $s<\bar{q}$. Moreover, it is readily checked that $(t+1)m-2\neq 0,\ m-1$. The first module in (3.18) is therefore zero in view of $A_{\bar{q}}$ and (3.4). It follows that

$$E_{tm+1}^{(h-1)m,\bar{q}-(h-1)m+1} = E_{tm}^{(h-1)m,\bar{q}-(h-1)m+1}, \qquad 1 \leqq t < h-1. \tag{3.19}$$

Using (3.7), (3.7)′, and (3.19), the identification (3.17) then follows.

We may also identify

$$E_2^{0,jq_0} = E_{(h-1)m}^{0,jq_0} \tag{3.20}$$

using (3.7), (3.7)′, and (3.10) (case i). If we apply (3.17) and (3.20) in (3.15) and define $z_{2j}=d_m^{-1}(x^{h-1}z_{2j-1})$ we see that (3.9) holds for $B_{\bar{q}+1}$.

It remains to prove that (3.14) and (3.15) are isomorphisms. Let $\bar{q}=jq_0+m-1$ and consider the sequence

$$0 \xrightarrow{d_m} E_m^{0,\bar{q}} \xrightarrow{d_m} E_m^{m,jq_0} \xrightarrow{d_m} E_m^{2m,jq_0-m+1}. \tag{3.21}$$

The last module is zero by (3.4) since by $A_{\bar{q}}$ we have $E_2^{0,jq_0-m+1}=0$. Thus to prove (3.14) an isomorphism it suffices to show that $E_{m+1}^{0,\bar{q}}=0$ and $E_{m+1}^{m,jq_0}=0$. The former follows using (3.7)′, (3.7)″, (3.10) (case ii), and acyclicity. To prove the latter consider the sequence

$$0 \xrightarrow{d_{tm}} E_{tm}^{m,jq_0} \xrightarrow{d_{tm}} E_{tm}^{(t+1)m,jq_0-tm+1}, \qquad 1 < t \leqq h-1. \tag{3.22}$$

If $t=h-1$ then $E_2^{(t+1)m,0}=0$. If $1<t<h-1$ then $jq_0-tm+1\not\equiv 0,\ m-1$, modulo q_0, and hence $E_2^{0,jq_0-tm+1}=0$ by $A_{\bar{q}}$. In either case the last module in (3.22) is therefore zero by (3.4) and hence $E_{tm}^{m,jq_0}=E_{tm+1}^{m,jq_0}$ for $1<t<h$. Combining this with (3.7)′, (3.7)″, and acyclicity it follows that $E_{m+1}^{m,jq_0}=0$. Finally, let $\bar{q}=jq_0$ and consider the sequence

$$0 \xrightarrow{d_{(h-1)m}} E_{(h-1)m}^{0,jq_0} \xrightarrow{d_{(h-1)m}} E_{(h-1)m}^{(h-1)m,(h-1)q_0+m-1} \xrightarrow{d_{(h-1)m}} E_{(h-1)m}^{2(h-1)m,(j-1)q_0+hm}. \tag{3.23}$$

The last module is evidently zero since $2(h-1)m>hm$. Moreover, $E_{(h-1)m+1}^{0,jq_0}=0$, $E_{(h-1)m+1}^{(h-1)m,(j-1)q_0+m-1}=0$ by (3.7)″ and acyclicity. Thus (3.23) reduces to the isomorphism (3.15). This completes the induction and A and B are proved.

Using A and B we shall now prove the following multiplication relations:

(3.24) $$z_1^2 = 0,$$

(3.25) $$z_1 z_{2j} = z_{2j+1},$$

(3.26) $$z_{2i} z_{2j} = (i, j) z_{2(i+j)}.$$

Note that z_1^2 has degree $2m-2$ and z_2 has degree $hm-2$. Thus if $h>2$, $z_1^2=0$. If $h=2$ put $z_1^2=tz_2$, $t\in A$; then by (3.8) and (3.9) respectively, we have

$$d_m(z_1^2) = xz_1 - z_1 x = xz_1 - xz_1 = 0, \qquad d_m(z_1^2) = txz_1.$$

Since xz_1 generates a free A-module, $t=0$ and (3.24) is established.

For $j=0$, (3.24) is trivial. Let $j>0$ and assume (3.25) for all $j'<j$. Put $z_1 z_{2j} = tz_{2j+1}$, $(t\in A)$; then using (3.8) we have

(3.27) $$d_m(z_1 z_{2j}) = xz_{2j} - z_1 d_m(z_{2j}).$$

If $h=2$ then using (3.9), the inductive assumption, and (3.24) in succession,

$$z_1 d_m(z_{2j}) = z_1(xz_{2j-1}) = xz_1 z_1 x_{2j-2} = 0.$$

If $h>2$, note that $d_m(z_{2j})\in E_2^{m,jq_0-m+1}$ which is zero by (3.4) and A. In either case (3.27) reduces to $d_m(z_1 z_{2j}) = xz_{2j}$. But by (3.8),

$$d_m(z_1 z_{2j}) = td_m(z_{2j+1}) = txz_{2j}.$$

Since xz_{2j} generates a free A-module, $t=1$ and (3.25) is proved.

For $i+j=0$, (3.26) is trivial. Let $i+j>0$ and assume (3.26) for all $i'+j'<i+j$. If we put $z_{2i}z_{2j} = t_{i,j} z_{2(i+j)}$, then using (3.9),

$$\begin{aligned} d_{(h-1)m}(z_{2i} z_{2j}) &= (x^{h-1} z_{2i-1}) z_{2j} + z_{2i}(x^{h-1} z_{2j-1}), \\ &= x^{h-1} z_1 (z_{2i-2} z_{2j} + z_{2i} z_{2j-2}), \\ &= x^{h-1} z_1 [(i-1, j) + (i, j-1)] z_{2i+2j-2}, \\ &= (i, j) x^{h-1} z_{2i+2j-1}. \end{aligned}$$

But also by (3.9),

$$d_{(h-1)m}(z_{2i} z_{2j}) = t_{i,j} d_{(h-1)m}(z_{2(i+j)}) = t_{i,j} x^{h-1} z_{2(i+j)-1}.$$

Since $x^{h-1} z_{2(i+j)-1}$ generates a free A-module, $t_{i,j} = (i, j)$.

The theorem now follows from (3.24), (3.25), and (3.26). For by (3.24), z_1 spans the subalgebra $\Lambda_A(z_1)$; by (3.26), the elements z_{2j} span the subalgebras $T_A(z_0, z_2, z_4, \cdots)$; and by (3.25), F is clearly isomorphic to the tensor product of the two subalgebras under the obvious map.

4. **Topological applications.** Let J denote a principal ideal domain of characteristic p (p is then zero or a prime).

THEOREM 2. *Let X be a topological space whose singular cohomology algebra $H^*(X, J) = J[x, h]$ where x is an element of even degree $m \geqq 2$. Let Ω denote the loop space of X at a base point x_0.*

(a) *$H^*(\Omega, J) \cong \Lambda_J(z_1) \otimes J[z_2, \infty, t]$, if $p=0$, where z_1 and z_2 have degree $m-1$ and $hm-2$, respectively, and the second factor is of binomial type.*

(b) *$H^*(\Omega, J) \cong \Lambda_J(z_1) \otimes_{i \geqq 0} J[z_{2p^i}, p, t^{(i)}]$, if p is prime, where z_1 and z_{2p^i} have degrees $m-1$ and $p^i(hm-2)$ respectively, and each $t^{(i)}$ is of binomial type.*

PROOF. Evidently X is arcwise connected, simply connected, and torsion-free. Associated with the Serre fibering[3] $f: E \to X$ where E is the space of paths beginning at x_0 (and Ω is the fibre at x_0) is a canonical spectral sequence of J-algebras (E_r) which is acyclic (since E is contractible) and such that $E_2^{p,q} \cong H^p(X, J) \otimes H^q(\Omega, J)$ (since X is a torsion-free and J is a principal ideal domain). These isomorphisms give an identification of the bigraded J-algebras, $E_2 = H^*(X, J) \otimes H^*(\Omega, J)$. Since the spectral sequence is initially decomposable, Theorem 2 follows immediately from Theorem 1 and its corollary.

Let P_n and Q_n denote the complex and quaternionic projective n-spaces, $1 \leqq n \leqq \infty$, respectively, and let C denote the Cayley plane. Their cohomology algebras are known to be

$H^*(P_n, J) = J[x, n+1]$, degree of $x = 2$;

$H^*(Q_n, J) = J[x, n+1]$, degree of $x = 4$;

$H^*(C, J) = J[x, 3]$, degree of $x = 8$.

Thus, Theorem 2 applies to P_n, Q_n, and C.

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[3] See reference [2, Chapter IV].