LACUNARY FOURIER SERIES ON NONCOMMUTATIVE GROUPS

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1. **Introduction.** In classical Fourier analysis lacunary series play a considerable role due to theorems of Kolmogoroff, Banach, Sidon and others. According to the usual definition a Fourier series $\sum a_k e^{in_k x}$ is called lacunary if $n_{k+1}/n_k > \lambda$ (>1) for all k. This definition makes use of the ordering of the integers and does not immediately extend to two dimensions or to more general groups which have been recognized as a proper setting for large parts of Fourier Analysis.

Let G be a compact group and as usual let $\hat{G}$ denote the set of equivalence classes of unitary irreducible representations of G. The set $\hat{G}$ has the following "hypergroup" structure: To each pair $\alpha, \beta \in \hat{G}$ there is attached a measure $\mu_{\alpha,\beta}$ on $\hat{G}$. This is determined by the decomposition of the Kronecker product $\alpha \otimes \beta$. In terms of this structure there is a natural duality between normal subgroups of G and certain subhypergroups of $\hat{G}$. Some of the abelian Pontrjagin duality extends to this situation, although two nonisomorphic finite groups G may have the same hypergroup structure of $\hat{G}$.

The purpose of this note is to point out how, in certain instances, the hypergroup structure of $\hat{G}$ is related to properties of Fourier expansions on G. In particular we give a definition of a lacunary Fourier series on G in terms of $\hat{G}$. If G is the circle group, our definition is formally quite different from the usual one but has similar implications. The definition is wide enough to cover the case of a series of the form $\sum a_n e^{ix_n}$, where x_n are independent variables and a well known theorem of Kolmogoroff about such series can be extended to Fourier series on the product $\prod_n U(n)$, $U(n)$ denoting the unitary group in n dimensions. Furthermore, the theorem of Banach stating that a lacunary L^1-series is an L^2-series is generalized to noncommutative groups.

2. **The duality.** We shall be concerned with compact groups G with normalized Haar measure dg and the familiar function spaces $L^1(G)$ and $L^2(G)$ of integrable and square integrable functions. The corresponding norms are denoted $\| \ \|_1$ and $\| \ \|_2$. Every function $f \in L^1(G)$ can be uniquely represented by a Fourier series

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Presented to the Society, October 27, 1956; received by the editors January 25, 1958.

$$f(g) \sim \sum_{\chi \in \hat{G}} d_\chi \operatorname{Tr} \langle A_\chi D_\chi(g) \rangle \tag{1}$$

where Tr denotes the usual trace, $\hat{G}$ is the set of equivalence classes of irreducible unitary representations of G, D_χ is a member of the class χ, d_χ is the degree of χ and A_χ is the linear transformation determined by

$$A_\chi = \int_G f(g) D_\chi(g^{-1}) dg. \tag{2}$$

For the expansion above, the Schur-Peter-Weyl formula is valid

$$\int_G |f(g)|^2 dg = \sum_{\chi \in \hat{G}} d_\chi \operatorname{Tr} \langle A_\chi A_\chi^* \rangle \tag{3}$$

finiteness of one side implying the finiteness of the other. (B^* denotes the adjoint of the operator B). In case f is a *central* function, that is f is invariant under inner automorphisms, the Fourier series (1) takes the form $f(g) \sim \sum a_\chi \chi(g)$ where $g \to \chi(g)$ is the character of the class χ.

The series (1) we call *absolutely convergent* if $\sum d_\chi^2 \|A_\chi\| < \infty$, $\| \ \|$ denoting the usual norm.

DEFINITION 2.1. A set S is called a (discrete) *hypergroup* if there is given a mapping $(\alpha, \beta) \to \mu_{\alpha,\beta}$ of $S \times S$ into the set of measures on S. A subset T of the hypergroup S is called a *subhypergroup* if all the measures $\mu_{\alpha,\beta}$ ($\alpha, \beta \in T$) have support contained in T.

If G is abelian, $\hat{G}$ is a group, and if G is nonabelian every tensor product $\alpha \otimes \beta$ for $\alpha, \beta \in \hat{G}$ has a direct decomposition into irreducible unitary components. This induces a hypergroup structure in $\hat{G}$. If A and B are two representations of G we call A and B *disjoint* if no irreducible component of A is equivalent to an irreducible component of B.

The identity transformation of arbitrary dimension will be called E and the irreducible unit representation of G will be called I. If M is an arbitrary subset of G we let $M^\perp$ stand for the set of classes $\alpha \in \hat{G}$ such that $D_\alpha(g) = E$ for each $g \in M$. Similarly if $\mathfrak{H} \in \hat{G}$ we let $\mathfrak{H}^\perp$ denote the subset of G determined by the equations $D_\alpha(g) = E$ for each $\alpha \in \mathfrak{H}$. For simplification we call a subhypergroup $\mathfrak{H}$ of $\hat{G}$ a *normal* subhypergroup if $I \in \mathfrak{H}$ and if $\alpha \in \mathfrak{H}$ implies $\bar{\alpha} \in \mathfrak{H}$ (the bar denotes complex conjugation).

We have then the following duality between normal subhypergroups of $\hat{G}$ and closed normal subgroups of G. This is closely related to a duality outlined by van Kampen [**3**].

THEOREM 1.

(i) *If* $M \subset G$, $M^\perp$ *is a normal subhypergroup of* $\hat{G}$ *and* $(M^\perp)^\perp$ *is the smallest closed normal subgroup of* G *containing* M.

(ii) *If* $\mathfrak{H} \subset \hat{G}$, $\mathfrak{H}^\perp$ *is a closed normal subgroup of* G *and* $(\mathfrak{H}^\perp)^\perp$ *is the smallest normal subhypergroup of* $\hat{G}$ *containing* $\mathfrak{H}$.

(iii) *If* N *is a closed normal subgroup of* G, $(G/N)^\frown = N^\perp$.

PROOF. (i) Let $\alpha, \beta \in M^\perp$ and let D_α, D_β be corresponding representations. From the direct decomposition of $D_\alpha \otimes D_\beta$ we get for the characters the decomposition

$$\alpha(g)\beta(g) = \chi_1(g) + \cdots + \chi_n(g) \tag{4}$$

where the χ_i are characters of irreducible representations whose dimensions d_i satisfy

$$d_\alpha \cdot d_\beta = d_1 + \cdots + d_n. \tag{5}$$

Now if $g \in M$ we have $\alpha(g) = d_\alpha$ and $\beta(g) = d_\beta$ and since $\max_g |\chi(g)| = d_\chi$ we conclude from (4) and (5) that $\chi_i(g) = d_i$ for all i. It follows, that $D_{\chi_i}(g) = E$ so $\chi_i \in M^\perp$ and $M^\perp$ is a subhypergroup which is normal. It is obvious that $(M^\perp)^\perp$ is a closed normal subgroup containing M. If N is some arbitrary closed normal subgroup containing M, then $(M^\perp)^\perp \subset N$ because if $n \in (M^\perp)^\perp - N$ we can (by going to the factor group G/N) find a representation $D \in \hat{G}$ such that $D(g) = E$ for $g \in N$ but $D(n) \neq E$. Then $D \in N^\perp - M^\perp$ which contradicts $M \subset N$.

(ii) If $\mathfrak{H} \subset \hat{G}$ it is clear that $\mathfrak{H}^\perp$ is a closed normal subgroup and $(\mathfrak{H}^\perp)^\perp$ is a normal subhypergroup. The matrix elements from $\mathfrak{H}$ and I can be regarded as a family of continuous functions on $G/\mathfrak{H}^\perp$ which separates points. Let $\mathfrak{R}$ be the set of linear combinations of matrix elements from the normal subhypergroup $\mathfrak{H}^*$ generated by $\mathfrak{H}$ and I. By the Peter-Weyl theorem $\mathfrak{R}$ is uniformly dense in the space of continuous functions on $G/\mathfrak{H}^\perp$. Now if there were a representation $D \in \mathfrak{H}^{\perp\perp} - \mathfrak{H}^*$, each matrix element $a(g)$ from D could be uniformly approximated by elements of $\mathfrak{R}$ but on the other hand $a(g)$ is orthogonal to $\mathfrak{R}$ by the orthogonality relations. This shows that $\mathfrak{H}^* = \mathfrak{H}^{\perp\perp}$.

(iii) Proof obvious.

3. Multipliers.

DEFINITION 3.1. A *hyperfunction* on $\hat{G}$ is a mapping which assigns to each $\chi \in \hat{G}$ a linear transformation of a complex vector space of dimension d_χ.

DEFINITION 3.2. A hyperfunction Γ on $\hat{G}$ is called a *multiplier* if for each Fourier series for a continuous function

$$f(g) \sim \sum d_\chi \operatorname{Tr} \langle A_\chi D_\chi(g) \rangle \tag{6}$$

the series

$$f_\Gamma(g) \sim \sum d_\chi \operatorname{Tr} \langle \Gamma_\chi A_\chi D_\chi(g) \rangle \tag{7}$$

is also a Fourier series for a continuous function.

It is easy to see from the closed graph theorem that if Γ is a multiplier there exists a bounded measure μ_Γ on G such that $f_\Gamma = f * \mu_\Gamma$ (convolution product). Less trivial is the following extension of a theorem of Sidon:

THEOREM 2. *Let Γ be a multiplier such that for each Fourier series* (6) (*with continuous f*) *the corresponding series* (7) *is absolutely convergent. Then there exists a function $F \in L^2(G)$ such that*

$$f_\Gamma = f * F \qquad \text{for all continuous } f.$$

PROOF. Let Γ be a multiplier with the properties stated in the theorem. Then $B\Gamma$ is also of that type provided B is a hyperfunction on $\hat{G}$ satisfying $\sup_\chi \|B_\chi\| < \infty$. To see that $B\Gamma$ really is a multiplier we remark that an absolutely convergent series (in the sense defined in this paper) is uniformly convergent on G; this last fact is easily verified by writing each Fourier matrix A_χ as $P_\chi V_\chi$ where P_χ is positive definite and V_χ is unitary. Using a previous remark we see that there exist bounded measures μ_Γ and $\mu_{B\Gamma}$ on G such that

$$f_{B\Gamma} = f * \mu_{B\Gamma},$$
$$f_\Gamma = f * \mu_\Gamma.$$

The hyperfunctions B satisfying $\sup_\chi \|B_\chi\| < \infty$ (the bounded hyperfunctions) form a Banach space under the norm $\sup_\chi \|B_\chi\|$. The mapping $T: B \rightarrow \mu_{B\Gamma}$ is a linear mapping of the Banach space of bounded hyperfunctions into the Banach space of measures on G and again from the closed graph theorem it follows easily that this mapping is continuous. Now an integrable function on G can be identified with a hyperfunction on $\hat{G}$ via the Fourier series expansion. If the function $\phi \in L^1(G)$ corresponds to B in this manner we see that $\mu_{B\Gamma}$ is absolutely continuous with respect to Haar measure and has a derivative, say $\phi^\Gamma \in L^1(G)$. Then the mapping $\tilde{T}: \phi \rightarrow \phi^\Gamma$ is a linear transformation of $L^1(G)$ into itself and since T above is continuous it follows that $\tilde{T}$ is *spectrally continuous* in the sense of [2]. Furthermore $\phi^\Gamma = \mu_\Gamma * \phi$ so $\tilde{T}$ commutes with right translations on G.

This being established, Theorem 2 follows from Theorem A in [2] which is an extension of a theorem of Littlewood and states that the spectrally continuous operators that commute with right translations are precisely the left convolutions with L^2-functions on G.

DEFINITION 3.3. A subset $S \subset \hat{G}$ is called *distinguished* if for every Fourier series for a continuous function

$$f(g) \sim \sum_{\chi \in \hat{G}} d_\chi \operatorname{Tr} \langle A_\chi D_\chi(g) \rangle \tag{8}$$

the subseries

$$\sum_{\chi \in S} d_\chi \operatorname{Tr} \langle A_\chi D_\chi(g) \rangle \tag{9}$$

also represents a continuous function f_S.

For abelian groups G, the distinguished sets were investigated in [1]. For the noncommutative case a partial description is given by

THEOREM 3. *The distinguished sets that preserve positivity in the sense that $f_S \geqq 0$ whenever $f \geqq 0$ are precisely the normal subhypergroups of $\hat{G}$.*

PROOF. The mapping $f \to f_S$ is continuous (uniform topology) by the closed graph theorem and commutes with left translations. Hence there exists a bounded measure μ_S on G such that $f_S = f * \mu_S$ for all f. The mapping $f \to f_S$ also commutes with right translations, so that the Fourier-Stieltjes series for μ_S has the form

$$\mu_S(g) \sim \sum_{\chi \in S} d_\chi \chi(g).$$

Using the assumption of the theorem we see that μ_S is a positive measure so by a theorem of Wendel [7] the condition $\mu_S * \mu_S = \mu_S$ implies that there exists a compact subgroup K of G such that $\mu_S(A) = \mu(A \cap K)$ for every Borel set A, where μ is the Haar measure on K. From the Fourier-Stieltjes series for μ_S we see that

$$\int_G \overline{D}_\chi(g) d\mu_S(g) = \int_K \overline{D}_\chi(k) d\mu(k) = \begin{cases} E & \text{if } \chi \in S, \\ 0 & \text{if } \chi \notin S \end{cases}$$

which implies $\overline{D}_\chi(k) = E$ for all k if and only if $\chi \in S$ or otherwise expressed: $K^\perp = S$.

On the other hand, if S is a normal subhypergroup then the Haar measure μ on $S^\perp$ extended to a measure on G by $\mu(E) = \mu(E \cap S^\perp)$ has the Fourier-Stieltjes series

$$\mu(g) \sim \sum_{\chi \in S} d_\chi \chi(g)$$

and for each continuous function f on G with Fourier series (8) the continuous function $f * \mu$ has Fourier series (9), proving that S is distinguished.

4. **Lacunary series.** In this section we discuss extensions of theorems of Kolmogoroff and Banach.

Let I be a set and to each element i of I attached an integer d_i. We consider the compact group $\mathbf{G}=\prod_{i\in I} U(d_i)$ where $U(m)$ denotes the unitary group in m dimensions. The projection D_i of $\mathbf{G}$ onto $U(d_i)$ is a unitary representation which clearly is irreducible, in other words I can be regarded as a subset of $\widehat{\mathbf{G}}$. We consider Fourier series of the form

$$\sum_{i\in I} d_i \operatorname{Tr}\langle A_i D_i(g)\rangle \tag{10}$$

and we shall now indicate the proof of the following theorem.

THEOREM 4. *Suppose $f\in L^1(\mathbf{G})$ and has a Fourier series of the form* (10). *Then $f\in L^2(\mathbf{G})$ and moreover $2^{-1/2}\|f\|_2\leqq\|f\|_1\leqq\|f\|_2$.*

In the case where $d_i=1$ for each $i\in I$, this result is a well known theorem of Kolmogoroff.

The essence of Theorem 4 is proved in [2]. In fact let us consider a finite subset J of I and a series of the form

$$s(g) = \sum_{j\in J} d_j \operatorname{Tr}\langle B_j D_j(g)\rangle. \tag{11}$$

It is then clear that

$$\int_G |s(g)|^m dg = \int_{V_J}\left|\sum_j d_j \operatorname{Tr}\langle B_j D_j\rangle\right|^m dD_J$$

where dD_J denotes the Haar measure on the product $V_J=\prod_{j\in J} U(d_j)$. By the proof of Lemma 4.1 in [2] and the relation (4.12) in [2] we get

$$\int_G |s(g)|^4 dg \leqq 2\left[\int_G |s(g)|^2 dg\right]^2.$$

Using the inequality

$$\left[\int |h(g)|\,dg\right]^2 \geqq \left[\int |h(g)|^4 dg\right]^{-1}\cdot\left[\int |h(g)|^2 dg\right]^3 \tag{12}$$

which is a special case of Hölder's inequality we obtain

$$\int_G |s(g)|\,dg \geqq 2^{-1/2}\left[\int_G |s(g)|^2 dg\right]^{1/2}. \tag{13}$$

By standard approximation arguments the function f can be approximated in L^1-norm by functions of the form (11), and (13) becomes valid for the function f. Theorem 4 is proved.

We shall now see that the set I in the above situation appears as a lacunary subset of $\hat{G}$ in a certain sense.

Let again G be an arbitrary compact group. If $\alpha, \beta \in \hat{G}$ we denote by D_α and D_β arbitrary members of α and β respectively, d_α and d_β the corresponding dimensions and $n_{\alpha,\beta}$ the number of irreducible components in $D_\alpha \otimes D_\beta$ (counted with multiplicity).

DEFINITION 4.1. A subset $S \subset \hat{G}$ is called *lacunary* if the two following conditions are satisfied.

(I) Whenever (α, β) and (γ, δ) are different pairs from S (that is, the characters $\alpha+\beta$ and $\gamma+\delta$ are different) $D_\alpha \otimes D_\beta$ and $D_\gamma \otimes D_\delta$ are disjoint.

(II) There exists a constant K such that $n_{\alpha,\beta} < K$ for all $\alpha, \beta \in S$.

A Fourier series of the form $\sum_{\chi \in S} d_\chi \operatorname{Tr} \langle A_\chi D_\chi(g) \rangle$ is called *lacunary* if S is lacunary.

The following statements show that the series (10) is indeed lacunary.

(i) $D_i \otimes D_j$ is irreducible if $i \neq j$.

(ii) If $d_i = 1$ then $D_i \otimes D_i$ is irreducible.

(iii) If $d_i \geqq 2$ then $D_i \otimes D_i$ decomposes into two irreducible parts (of dimensions $(d_i^2 + d_i)/2$ and $(d_i^2 - d_i)/2$).

(iv) $D_i \otimes D_i$ is disjoint from $D_j \otimes D_j$ if $i \neq j$.

(i) and (ii) are obvious. (iii) is a corollary of Lemma 4.1 in [2] combined with the fact that the space of symmetric and antisymmetric tensors are left invariant by $D_i \otimes D_i$. Concerning (iv) we remark that the number of irreducible components common to $D_i \otimes D_i$ and $D_j \otimes D_j$ is equal to

$$\int_G (\chi_i^2 \bar{\chi}_j^2) dg = \int_{U(d_i) \times U(d_{ij})} (\operatorname{Tr} D_i)^2 (\operatorname{Tr} D_j^{-1})^2 dD_i dD_j.$$

(dD_m is the Haar measure on $U(d_m)$), and this last integral vanishes as shown in the proof of the cited lemma.

For a general compact group we have a simple result in similar direction.

THEOREM 5. *Let f be a central function in $L^1(G)$ and suppose f has a lacunary Fourier series. Then $f \in L^2(G)$.*

PROOF. The Fourier expansion of f can be written $f(g) \sim \sum_{\chi \in S} a_\chi \chi(g)$ where a_χ is a complex number and S is lacunary. We consider a finite partial sum $s(g) = \sum_1^N a_n \chi_n(g)$. Then

$$s^2 = \sum a_p^2\chi_p^2 + 2 \sum_{p>q} a_p a_q \chi_p \chi_q.$$

If we here expand χ_p^2 and $\chi_p\chi_q$ into a sum of characters the same χ_i will not occur more than once due to condition (I), and we get easily

$$\int_G |s(g)|^4 dg = \sum_1^N |a_p|^4 n_{p,p} + 2 \sum_{p>q} |a_p|^2 |a_q|^2 n_{p,q}$$
$$\leqq K\left(\sum_1^N |a_p|^2\right)^2.$$

Using the inequality (12) we obtain the conclusion $f \in L^2(G)$ in exactly the same manner as before.

Concerning the relation with the classical definition of lacunary series we remark that a series of the form $\sum a_k e^{in_k x}$ where $n_{k+1}/n_k > 2$ for all k, is lacunary in the sense of Definition 4.1. This is easily verified and is indeed a basic property in the proof of most classical theorems on lacunary series.

As a simple consequence of Theorem 5 we mention the following fact:

THEOREM 6. *Let G be an abelian group which is compact and not totally disconnected. Suppose the infinite series*

$$\sum a_\chi \sigma\cdot\chi(g) \tag{14}$$

is a Fourier series for some L^1-function for each permutation σ of $\hat{G}$. Then $\sum_{\chi\in\hat{G}} |a_\chi|^2 < \infty$.

PROOF. By a theorem of Pontrjagin $\hat{G}$ has an infinite cyclic subgroup and therefore an infinite countable lacunary subset. Now we can write

$$\sum_{a_\chi\neq 0} a_\chi\chi(g) = \sum_S a_\chi\chi(g) + \sum_T a_\chi\chi(g)$$

where both sets S and T are infinite and $\sum_S |a_\chi| < \infty$. Hence $\sum_T a_\chi\sigma\cdot\chi(g)$ is an L^1-series for every permutation σ of $\hat{G}$. Choose this permutation in such a way that $\sum_T a_\chi\sigma\cdot\chi(g)$ is a lacunary series and apply Theorem 5. Q.E.D.

If G is an arbitrary compact group, $\hat{G}$ need not possess any infinite lacunary subsets. As a simple example we mention $G = SU(2)$. The hypergroup structure of $\hat{G}$ is here described by the Clebsch-Gordan formula [8] which shows that the requirement (II) in Definition 4.1 is not fulfilled for any infinite subset $S \subset \hat{G}$.

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