

LEMMA 2'. Let L be a distributive lattice with 0 element and with finite bounded chains. L_{\cup} may be imbedded in the discrete cardinal product of as many copies of L as there exist join-irreducible elements in L .

THEOREM 3'. Let L be a lattice with 0 element and with finite bounded chains. All join-endomorphisms of L form a distributive lattice if and only if L is distributive.

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LINEAR COMPLETENESS AND HYPERBOLIC TRIGONOMETRY

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In this paper we show the uniqueness of the relation between a segment and its angle of parallelism as derived from a model. Upon generalizing this relation hyperbolic trigonometry follows in a remarkably simple way.

To introduce proper terminology [5, pp. 11–28] let Σ denote an *axiom system*, that is a certain set of axioms together with the undefined technical and logical or universal terms used to state the axioms. We define the terms *interpretation* and *model* in the usual fashion. It is useful to make a clear distinction between the following three concepts. (1) A Σ -*statement* is a meaningful expression, not necessarily true, in the technical and universal terms of Σ . (2) A T - Σ -*statement* is a true Σ -statement in the sense of being logically derivable from Σ . (3) If I denotes an interpretation of Σ , then an I - Σ -*statement* is a Σ -statement holding for the model which results from the interpretation I .

For the purpose of this paper let Σ be the postulate system of Hilbert [3, pp. 2–30] with the Euclidean axiom of parallelism replaced by the characteristic postulate of hyperbolic plane geometry [6, p. 66]. Some authors have used models to find I - Σ -statements [1, §39–117; 2]. Such a procedure, however, may be objectionable [1, §118]. Conceivably an I - Σ -statement could be made which is not a T - Σ -statement, but is merely a property of a particular model. In other words, it might be possible to find contradictory I - Σ -statements in two different models. Clearly, if this happens it indicates that our system Σ is not complete [5, pp. 33–36]. Any I - Σ -statement that is not a T - Σ -statement would still be compatible with the axioms of Σ .

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The special Σ -statement to be considered here is the relation between a distance and its corresponding angle of parallelism [4, pp. 143–144]. To avoid ambiguity we assign the unit of length to a segment whose angle of parallelism is $2 \operatorname{arc} \tan e^{-1}$. Assume now that for a prescribed segment, not of length one, two different angles of parallelism are found in two models. This would be the situation, suggested above, of two contradictory I - Σ -statements. We propose to show that two different formulas for the angle of parallelism are impossible in a geometry based on Σ . To this end, let x denote the given segment perpendicular to a line MN , θ the smaller, and θ' the larger of the angles of parallelism. If θ' were the true angle, lines passing through the proper end point of x and making angles greater than or equal to θ and less than θ' with x , would intersect MN [6, p. 67]. The line MN would then have more points than in the case of θ being the angle of parallelism. This contradicts the postulate of linear completeness [3, p. 30]. Hence the conflicting I - Σ -statements on the angle of parallelism cannot both be compatible with Σ . The functional relationship between x and θ must be unique and the same in all models. It is consistent with the axioms of Σ ; whether it is a T - Σ -statement, that is provable from these axioms without additional assumptions is not of concern to us here.

The relationship in question derived from two different, though closely related, models [1, §74; 6, pp. 214–216] is $e^{-x} = \tan \theta/2$. We admit obtuse angles for x negative [6, p. 77] for the purpose of the present paragraph. In order to generalize this formula we consider two parallel lines which intersect a third line in two points P , Q such that distance $PQ = z$. Let θ and ϕ be the oblique angles in the triangle determined by P , Q , and the ideal point of the given parallels [6, pp. 71–75]. The distances corresponding to these angles, regarded as angles of parallelism, are denoted by x and y . There is exactly one line, perpendicular to PQ and parallel to the first one of the given parallel lines. Its distance from P is equal to x . Because of the transitivity of parallelism [4, p. 139] this line perpendicular to PQ is also parallel to the second one of the given parallels and its distance from Q is necessarily y . Since $z = x + y$, we conclude that

$$(1) \quad e^{-z} = \tan \theta/2 \tan \phi/2.$$

In (1) z is positive for the angles chosen. Relation (1) is readily changed to

$$(2) \quad \cosh z = \frac{1 + \cos \theta \cos \phi}{\sin \theta \sin \phi}.$$

At this point let the customary notation apply to a right triangle ABC . Through the vertex B we draw the two parallels to line AC and designate by θ the acute angle of parallelism corresponding to side a . Using (2) with respect to the hypotenuse c and the parallels in either sense we obtain

$$(3) \quad \cosh c = \frac{1 + \cos \alpha \cos (\theta + \beta)}{\sin \alpha \sin (\theta + \beta)},$$

$$(4) \quad \cosh c = \frac{1 - \cos \alpha \cos (\theta - \beta)}{\sin \alpha \sin (\theta - \beta)}.$$

Equating (3) and (4) we have $\cos \alpha \sin \theta = \sin \beta$. By means of a suitable formula for the angle of parallelism [6, p. 151] this takes the form

$$(5) \quad \cos \alpha \operatorname{sech} a = \sin \beta.$$

Hence by analogy,

$$(6) \quad \cos \beta \operatorname{sech} b = \sin \alpha.$$

We now expand (3), use formulas for θ , and apply (5). Thus,

$$(7) \quad \cosh c = \cosh a \cosh b.$$

It is easily seen that (5), (6), (7) allow us to derive the remaining formulas for the right triangle.

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