ON A TAUBERIAN THEOREM OF LANDAU

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1. **Introduction.** It was the plan of Tschebyschef [1] and Sylvester [2] to deduce the prime number theorem[1]

$$\psi(x) \sim x$$

from the formula

$$T(x) = \sum_{n \leq x} \psi\left(\frac{x}{n}\right) = \log [x]!$$

However they succeeded only in proving the existence of positive constants c_1 and c_2 such that

$$c_1 x < \psi(x) < c_2 x.$$

Landau discusses this problem in his Handbuch [3, pp. 79–83], where he proves that if $A(x)$ is any monotone nondecreasing function of x such that

$$T(x) = \sum_{n \leq x} A\left(\frac{x}{n}\right) = x \log x + bx + o(x)$$

with b a constant, then

$$c_1 x < A(x) < c_2 x, \tag{1}$$

where c_1 and c_2 are positive constants. This gives Tschebyschef's theorem if we take

$$A(x) = \psi(x)$$

and use the fact that

$$\log [x]! = x \log x - x + O(\log x).$$

Landau states that if only the condition

$$T(x) = x \log x + O(x)$$

is assumed, then (1) does not follow, a remark whose incorrectness was proved by H. N. Shapiro [4]. Landau then goes on to prove [3, pp. 598–604] that if

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[1] Here and in the sequel, $\psi(x)$, $\Lambda(n)$, $\mu(n)$, and $\theta(x)$ denote the usual functions of prime number theory.

$$T(x) = x \log x + bx + o\left(\frac{x}{\log^2 x}\right),$$

then $A(x) \sim x$. The proof requires the formula

$$\sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n} = -1, \tag{2}$$

and since this is "deeper" than the prime number theorem, he concludes that Tschebyschef was foredoomed to failure. In the present paper we shall correct this statement by showing that a slightly stronger theorem than Landau's can be obtained using only the prime number theorem. The theorem to be proved is

THEOREM 1. *If $A(x)$ is nondecreasing, and*

$$T(x) = x \log x + bx + o\left(\frac{x}{\log x}\right), \tag{3}$$

then $A(x) \sim x$.

It should be mentioned that the best theorem in this direction has been obtained by A. E. Ingham [**5**], but the proof makes use of Wiener's Tauberian theorem, and therefore cannot be considered elementary.

2. **A Selberg-like formula.** We shall require the following elementary lemma:

LEMMA 1. *If $F(x)$ is any function defined for $x \geqq 1$, and if*

$$G(x) = \log x \sum_{n \leqq x} F\left(\frac{x}{n}\right),$$

then

$$F(x) \log x + \sum_{n \leqq x} F\left(\frac{x}{n}\right) \Lambda(n) = \sum_{n \leqq x} \mu(n) G\left(\frac{x}{n}\right).$$

The proof is a straight-forward application of the Möbius inversion formula. For details, see Trost [**6**, p. 66]. Now let $A(x)$ be a function for which the hypothesis (3) is satisfied, and apply Lemma 1 with $F(x) = A(x)$. The corresponding $G(x)$ is

$$G(x) = \log x \sum_{n \leqq x} A\left(\frac{x}{n}\right) = T(x) \log x = x(\log x)^2 + bx \log x + o(x),$$

and so by Lemma 1,

$$
\tag{4}
\begin{aligned}
&A(x)\log x+\sum_{n\le x}A\left(\frac{x}{n}\right)\Lambda(n)\\
&\quad=\sum_{n\le x}\mu(n)\frac{x}{n}\left(\log\frac{x}{n}\right)^2+\sum_{n\le x}\mu(n)b\frac{x}{n}\log\frac{x}{n}+\sum_{n\le x}\mu(n)o\left(\frac{x}{n}\right).
\end{aligned}
$$

It is well known that the right side of (4) equals $2x\log x+o(x\log x)$ so that we obtain

THEOREM 2. *If $A(x)$ satisfies condition* (3), *then*

$$
\tag{5}
A(x)\log x+\sum_{n\le x}A\left(\frac{x}{n}\right)\Lambda(n)=2x\log x+o(x\log x).
$$

It can be seen that this result is very similar to the Selberg formula for $\theta(x)$ with a weaker error term (cf. [**7**], especially the remark on p. 313).

3. **Proof of the main theorem.** We shall now derive Theorem 1 from the identity (5), and it is here that the monotonicity of $A(x)$ becomes important. Note first that (5) and the fact that $A(x)$ is nondecreasing imply that $A(x)>0$ for all sufficiently large x. This fact together with (5) then yields that for $x=y+o(x)$ we have $A(x)=A(y)+o(x)$. Thus it is easily seen that without loss of generality we may assume that for all $x>0$, we have $A(x)=A([x])$. Since $A(x)$ is nondecreasing we can then write

$$
A(x)=\sum_{m\le x}a_m,
$$

where $a_m\geqq 0$. We can then rewrite (5) as

$$
A(x)\log x+\sum_{mn\le x}a_m\Lambda(n)=2x\log x+o(x\log x),
$$

or equivalently

$$
\tag{6}
A(x)\log x+\sum_{m\le x}a_m\psi\left(\frac{x}{m}\right)=2x\log x+o(x\log x).
$$

From the prime number theorem and the result of [**4**] (in particular Equation (6) therein) it follows that

$$
\sum_{m\le x}a_m\psi\left(\frac{x}{m}\right)=(1+o(1))x\sum_{m\le x}\frac{a_m}{m}\sim x\log x,
$$

which together with (6) above yields $A(x)\sim x$, the desired result.

The following application of Theorem 1 is interesting. Let $A(x)=[x]+M(x)$, where $M(x)=\sum_{n\le x}\mu(n)$. Since $1+\mu(n)\geqq 0$, $A(x)$ is monotone. The corresponding $T(x)$ is

$$\begin{aligned} T(x) &= \sum_{n \leq x}\left(\left[\frac{x}{n}\right] + M\left(\frac{x}{n}\right)\right) \\ &= x \log x + cx + O(x^{1/2}) + \sum_{n \leq x} \mu(n)\left[\frac{x}{n}\right] \\ &= x \log x + cx + O(x^{1/2}) \end{aligned}$$

since $\sum_{n \leq x} \mu(n)[x/n] = 1$ (cf. [**3**, pp. 576–577]). Thus the hypothesis of Theorem 1 is satisfied and we conclude that $A(x) \sim x$, which is here the same as asserting $M(x) = o(x)$. The above thus provides another proof of the result of Landau [**8**] which asserts that $\psi(x) \sim x$ implies $M(x) = o(x)$.

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