

ON A TAUBERIAN THEOREM OF LANDAU

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1. **Introduction.** It was the plan of Tschebyschef [1] and Sylvester [2] to deduce the prime number theorem¹

$$\psi(x) \sim x$$

from the formula

$$T(x) = \sum_{n \leq x} \psi\left(\frac{x}{n}\right) = \log [x]!$$

However they succeeded only in proving the existence of positive constants c_1 and c_2 such that

$$c_1x < \psi(x) < c_2x.$$

Landau discusses this problem in his Handbuch [3, pp. 79–83], where he proves that if $A(x)$ is any monotone nondecreasing function of x such that

$$T(x) = \sum_{n \leq x} A\left(\frac{x}{n}\right) = x \log x + bx + o(x)$$

with b a constant, then

$$(1) \quad c_1x < A(x) < c_2x,$$

where c_1 and c_2 are positive constants. This gives Tschebyschef's theorem if we take

$$A(x) = \psi(x)$$

and use the fact that

$$\log [x]! = x \log x - x + O(\log x).$$

Landau states that if only the condition

$$T(x) = x \log x + O(x)$$

is assumed, then (1) does not follow, a remark whose incorrectness was proved by H. N. Shapiro [4]. Landau then goes on to prove [3, pp. 598–604] that if

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¹ Here and in the sequel, $\psi(x)$, $\wedge(n)$, $\mu(n)$, and $\theta(x)$ denote the usual functions of prime number theory.

$$T(x) = x \log x + bx + o\left(\frac{x}{\log^2 x}\right),$$

then $A(x) \sim x$. The proof requires the formula

$$(2) \quad \sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n} = -1,$$

and since this is “deeper” than the prime number theorem, he concludes that Tschebyschef was foredoomed to failure. In the present paper we shall correct this statement by showing that a slightly stronger theorem than Landau’s can be obtained using only the prime number theorem. The theorem to be proved is

THEOREM 1. *If $A(x)$ is nondecreasing, and*

$$(3) \quad T(x) = x \log x + bx + o\left(\frac{x}{\log x}\right),$$

then $A(x) \sim x$.

It should be mentioned that the best theorem in this direction has been obtained by A. E. Ingham [5], but the proof makes use of Wiener’s Tauberian theorem, and therefore cannot be considered elementary.

2. A Selberg-like formula. We shall require the following elementary lemma:

LEMMA 1. *If $F(x)$ is any function defined for $x \geq 1$, and if*

$$G(x) = \log x \sum_{n \leq x} F\left(\frac{x}{n}\right),$$

then

$$F(x) \log x + \sum_{n \leq x} F\left(\frac{x}{n}\right) \Lambda(n) = \sum_{n \leq x} \mu(n) G\left(\frac{x}{n}\right).$$

The proof is a straight-forward application of the Möbius inversion formula. For details, see Trost [6, p. 66]. Now let $A(x)$ be a function for which the hypothesis (3) is satisfied, and apply Lemma 1 with $F(x) = A(x)$. The corresponding $G(x)$ is

$$G(x) = \log x \sum_{n \leq x} A\left(\frac{x}{n}\right) = T(x) \log x = x(\log x)^2 + bx \log x + o(x),$$

and so by Lemma 1,

$$(4) \quad A(x) \log x + \sum_{n \leq x} A\left(\frac{x}{n}\right) \Lambda(n) \\ = \sum_{n \leq x} \mu(n) \frac{x}{n} \left(\log \frac{x}{n}\right)^2 + \sum_{n \leq x} \mu(n) b \frac{x}{n} \log \frac{x}{n} + \sum_{n \leq x} \mu(n) o\left(\frac{x}{n}\right).$$

It is well known that the right side of (4) equals $2x \log x + o(x \log x)$ so that we obtain

THEOREM 2. *If $A(x)$ satisfies condition (3), then*

$$(5) \quad A(x) \log x + \sum_{n \leq x} A\left(\frac{x}{n}\right) \Lambda(n) = 2x \log x + o(x \log x).$$

It can be seen that this result is very similar to the Selberg formula for $\theta(x)$ with a weaker error term (cf. [7], especially the remark on p. 313).

3. Proof of the main theorem. We shall now derive Theorem 1 from the identity (5), and it is here that the monotonicity of $A(x)$ becomes important. Note first that (5) and the fact that $A(x)$ is nondecreasing imply that $A(x) > 0$ for all sufficiently large x . This fact together with (5) then yields that for $x = y + o(x)$ we have $A(x) = A(y) + o(x)$. Thus it is easily seen that without loss of generality we may assume that for all $x > 0$, we have $A(x) = A([x])$. Since $A(x)$ is nondecreasing we can then write

$$A(x) = \sum_{m \leq x} a_m,$$

where $a_m \geq 0$. We can then rewrite (5) as

$$A(x) \log x + \sum_{mn \leq x} a_m \Lambda(n) = 2x \log x + o(x \log x),$$

or equivalently

$$(6) \quad A(x) \log x + \sum_{m \leq x} a_m \psi\left(\frac{x}{m}\right) = 2x \log x + o(x \log x).$$

From the prime number theorem and the result of [4] (in particular Equation (6) therein) it follows that

$$\sum_{m \leq x} a_m \psi\left(\frac{x}{m}\right) = (1 + o(1))x \sum_{m \leq x} \frac{a_m}{m} \sim x \log x,$$

which together with (6) above yields $A(x) \sim x$, the desired result.

The following application of Theorem 1 is interesting. Let $A(x) = [x] + M(x)$, where $M(x) = \sum_{n \leq x} \mu(n)$. Since $1 + \mu(n) \geq 0$, $A(x)$ is

monotone. The corresponding $T(x)$ is

$$\begin{aligned} T(x) &= \sum_{n \leq x} \left(\left[\frac{x}{n} \right] + M\left(\frac{x}{n}\right) \right) \\ &= x \log x + cx + O(x^{1/2}) + \sum_{n \leq x} \mu(n) \left[\frac{x}{n} \right] \\ &= x \log x + cx + O(x^{1/2}) \end{aligned}$$

since $\sum_{n \leq x} \mu(n) [x/n] = 1$ (cf. [3, pp. 576–577]). Thus the hypothesis of Theorem 1 is satisfied and we conclude that $A(x) \sim x$, which is here the same as asserting $M(x) = o(x)$. The above thus provides another proof of the result of Landau [8] which asserts that $\psi(x) \sim x$ implies $M(x) = o(x)$.

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