A CHARACTERIZATION OF LIGHT OPEN MAPS OF EUCLIDEAN SPACES INTO EUCLIDEAN SPACES[1]

JAMES J. ANDREWS

We will use the term open map to mean a map f of X into Y such that $f(U)$ is open in Y for every open set U in X. A map f of R^n (Euclidean n-space) into R^m is pseudo-monotone if and only if $R^n-f^{-1}(X)$ has no bounded component for every closed set X in R^n such that R^m-X has no bounded component.

The purpose of this note is to prove that a light map f of R^n into R^m is open if and only if f is pseudo-monotone. But first we must prove the following lemma:

LEMMA. *If f is an open map of R^n into R^m, U is a bounded open set in R^n, and $f(U)\cap W\neq\emptyset$ for some component W of R^m-f(bdry U), then $W\subset f(U)$. Thus, in particular, $f(U)$ does not intersect the unbounded component of R^m-f(bdry U).*

PROOF. Assume $W\not\subset f(U)$. Let p be a point of $W-f(U)$ and q be a point of $W\cap f(U)$. Let $\bar{pq}$ be an arc from p to q in R^n-f(bdry U). Since q is in the compact set $f(\overline{U})$ and p is not, there is a first point x on the arc $\bar{pq}$ in the order p to q such that x is in $f(\overline{U})$. We note x is not in f(bdry U) since $\bar{pq}\cap f$(bdry $U)=\emptyset$. Therefore $f^{-1}(x)\cap U\neq\emptyset$ and $x\subset$bdry $f(U)$. Hence f is not open.

REMARK. If W is the union of X and the bounded components of R^m-X, then R^m-W has no bounded components.

THEOREM. *A light map f of R^n into R^m is open if and only if f is pseudo-monotone.*

PROOF. Assume f is open but not pseudo-monotone. Then by definition of pseudo-monotone there is a closed set X in R^m such that R^m-X has no bounded component and $R^n-f^{-1}(X)$ has a bounded component. Let $R^n-f^{-1}(X)=A\cup B$ where A and B are open, $A\cap B=\emptyset$, and A is both bounded and nonempty. For the open set A we have f(bdry $A)\subset X$ and $f(A)\subset R^m-X$. Hence $f(A)\cap W\neq\emptyset$ for some unbounded component W of R^m-X. (R^m-X has no bounded component.) But $R^m-X\subset R^m-f$(bdry A). Therefore $f(A)\cap W'\neq\emptyset$,

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where W' is the unbounded component of $R^m - f(\text{bdry } A)$ containing W. By our lemma this is not possible.

Assume f is pseudo-monotone but not open. Then there is an open set U in R^n such that $f(U)$ is not open in R^m. Let p be a point of $f(U) \cap \text{bdry} f(U)$, let q be a point of $f^{-1}(p) \cap U$, and let V be a bounded neighborhood of q lying in U such that the boundary of V does not intersect the 0-dimensional set $f^{-1}(p)$. The boundary of $V = B$ is compact, hence there is a spherical neighborhood S of p which does not intersect the compact set $f(B)$. Let x be a point of the set $S - f(U)$. $S - f(U)$ is not empty because p is the boundary of $f(U)$. Let b be the first point on the line segment from x to p which is in the compact set $f(\overline{V})$. Then b is in $f(V)$ since b is in S and $S \cap f(\text{bdry } V) = \emptyset$. Let a be a point of $f^{-1}(b) \cap V$. Let r denote the ray from p through x and let r' denote the subray of r with source at x. Let S' denote the open set $S - r'$. Let V' be a neighborhood of a, whose closure lies in the open set $f^{-1}(S) \cap V$, and whose boundary does not intersect the 0-dimensional set $f^{-1}(b)$. b is in $f(V') \cap r$ and is the only point of the ray $r' \cup \bar{x}\bar{b}$ which lies in $f(\overline{V}')$. Hence b is in an unbounded component of $R^m - f(\text{bdry } V')$. Let W be the union of $f(\text{bdry } V')$ and the bounded components of $f(\text{bdry } V')$, then $R^m - W$ has no bounded components. Consider the set $f^{-1}(W)$. $a \notin f^{-1}(W)$ because $b \notin W$. But bdry $V' \subset f^{-1}(W)$; hence the bounded set $V' \subset f^{-1}(W)$ and therefore $a \in f^{-1}(W)$.

In showing that an open map is pseudo-monotone we did not use the fact that f was light. Hence an open map of R^n into R^m is pseudo-monotone.

In comparing the theorem with a result of A. D. Wallace [2], the referee points out that the following proposition is false:

PROPOSITION. *If f is a pseudo-monotone map of R^n into R^m, then there exists an increasing sequence C_1, C_2, $\cdots$ of n-cells such that $UC_i = R^n$ and f/C_i is quasi-monotone.*

BIBLIOGRAPHY

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2. A. D. Wallace, *Quasi-monotone transformations*, Duke Math. J. vol. 7 (1940) pp. 136–145.

UNIVERSITY OF GEORGIA