1. Introduction. A homothetic mapping dilates any figure into a similar figure; it is therefore conformal. T. Sumitomo [1] proved that in a positive definite, compact orientable Riemann space such a mapping either does not exist or reduces to a motion; that is, the dilation constant is zero. The main purpose of this note is to prove Theorem 2.1 which gives some group theoretic insight into the reason for the scarcity of homothetic mappings. The technique used is that of Lie differentiation [2; 3].

2. Homothetic mappings. Let $V_n$ be a Riemann space with fundamental metric $g_{ij}(x)$. Let $\xi^i(x)$ be a vector field defining a one parameter Lie group and $L$ be the symbol of Lie differentiation based on $\xi^i(x)$. The condition that $\xi^i$ define a homothetic mapping of $V_n$ is

$$Lg_{ij} = \xi_{i,j} + \xi_j,i = cg_{ij}$$

where $\xi_{i,j}$ is the covariant derivative of $\xi_i$ and $c$ is a constant $\neq 0$. (If $c=0$ (2.1) are Killing's equations defining a motion.) Let $G_{r+1}$ be the full Lie group of homothetic mappings of $V_n$, the vectors of the group being $\xi_{(\alpha)}$, $\alpha=1, 2, \cdots, r+1$, each vector defining a partial Lie derivative $L_\alpha$. Then $L_\alpha g_{ij} = c_\alpha g_{ij}$ and any vector $k^\alpha \xi_{(\alpha)}$ defines the mapping such that

$$Lg_{ij} = k^\alpha L_\alpha g_{ij} = k^\alpha c_\alpha g_{ij}.$$  

Now $c_\alpha$ are constants (not all=0) so that $k^\alpha c_\alpha = 0$ admits $r$ independent solutions $k_\sigma^{(\alpha)}$, $\sigma = 1, \cdots, r$. Hence without loss of generality we may choose as a basis the vectors

$$\xi_{(\alpha)} = k_\sigma^{(\alpha)} \xi_{(\sigma)}, \quad \eta = \xi_{(r+1)}$$

so that $L_{(\alpha)}g_{ij} = 0$, $L_{r+1}g_{ij} = c g_{ij}$. This shows that $\xi_{(\alpha)}$ define a $G_r$ group of motions of $V_n$. We prove that this $G_r$ is an invariant subgroup of $G_{r+1}$. To this end let $\xi^i$ be a vector of $G_r$; we must show that if $X = \xi^i \partial/\partial x^i$ and $Y = \eta^i \partial/\partial x^i$ then $(X, Y) = c X_\sigma$. Let $(X, Y)f = Zf = \xi^i \partial f/\partial x^i$; then $\xi^i = \eta^i \delta_{i,h} - \delta^h \xi^i$. We also know that $\xi^i$ and $\eta^i$ define affine collineations of $V_n$ so that $\xi^i_{,ik} + \xi^h R^i_{jkh} = \eta^i_{,ik} + \eta^h R^i_{jkh} = 0$ so that
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\[ \xi_{,j} = \xi_{,h} + \eta_{,j} \xi_{,h} - \eta \xi_{,h} R_{hj} + \eta \xi R_{hj} \]

or

\[ \xi_{,j} = \xi_{,h} + \eta_{,j} \xi_{,h} + \xi \eta R_{hj} \]

Interchanging \( k \) and \( j \) and adding the two sets of equations and making use of (2.1), we find that

\[ \xi_{,k} + \xi_{,j,k} = g^{hl}(\eta_{,h} + \eta_{,h,k}) + g^{hl}\xi_{,k} R^{hj} \]

so that \( \xi \) defines a motion and is therefore in \( G_r \). Thus we have

**Theorem 2.1.** The full group \( G_{r+1} \) of homothetic mappings of \( V_n \) contains an invariant subgroup \( G_r \) of motions and a \( G_1 \) subgroup of dilations.

Another theorem that may be proved in a similar manner is

**Theorem 2.2.** A Riemann space of constant nonzero curvature does not admit any homothetic mappings except motions.

This is a generalization of some theorems of Sumitomo, since it is independent of compactness or orientability.

**Bibliography**


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