HOMOTHETIC MAPPINGS OF RIEMANN SPACES

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1. **Introduction.** A homothetic mapping dilates any figure into a similar figure; it is therefore conformal. T. Sumitomo [1] proved that in a positive definite, compact orientable Riemann space such a mapping either does not exist or reduces to a motion; that is, the dilation constant is zero. The main purpose of this note is to prove Theorem 2.1 which gives some group theoretic insight into the reason for the scarcity of homothetic mappings. The technique used is that of Lie differentiation [2; 3].

2. **Homothetic mappings.** Let V_n be a Riemann space with fundamental metric $g_{ij}(x)$. Let $\xi^i(x)$ be a vector field defining a one parameter Lie group and L be the symbol of Lie differentiation based on $\xi^i(x)$. The condition that ξ^i define a homothetic mapping of V_n is

$$(2.1)\qquad Lg_{ij} \equiv \xi_{i,j} + \xi_{j,i} = cg_{ij}$$

where $\xi_{i,j}$ is the covariant derivative of ξ_i and c is a constant $\neq 0$. (If $c=0$ (2.1) are Killing's equations defining a motion.) Let G_{r+1} be the full Lie group of homothetic mappings of V_n, the vectors of the group being $\xi^i{}_{(\alpha)}$, $\alpha=1, 2, \cdots, r+1$, each vector defining a partial Lie derivative L_α. Then $L_\alpha g_{ij}=c_\alpha g_{ij}$ and any vector $k^\alpha\xi^i_\alpha$ defines the mapping such that

$$(2.2)\qquad Lg_{ij} = k^\alpha L_\alpha g_{ij} = k^\alpha c_\alpha g_{ij}.$$

Now c_α are constants (not all $=0$) so that $k^\alpha c_\alpha=0$ admits r independent solutions $k^\alpha_{(\sigma)}$, $\sigma=1, \cdots, r$. Hence without loss of generality we may choose as a basis the vectors

$$(2.3)\qquad \bar{\xi}^i_{(\sigma)} = k^\alpha_{(\sigma)}\xi^i_{(\alpha)}, \qquad \eta^i = \xi^i_{(r+1)},$$

so that $\overline{L}_{(\sigma)}g_{ij}=0$, $L_{r+1}g_{ij}=cg_{ij}$. This shows that $\bar{\xi}^i_{(\sigma)}$ define a G_r group of motions of V_n. We prove that this G_r is an invariant subgroup of G_{r+1}. To this end let ξ^i be a vector of G_r; we must show that if $X=\xi^i\partial/\partial x^i$ and $Y=\eta^i\partial/\partial x^i$ then $(X, Y)=c^\sigma X_a$. Let $(X, Y)f=Zf=\zeta^i\partial f/\partial x^i$; then $\zeta^i=\xi^h\eta^i{}_{,h}-\eta^h\xi^i{}_{,h}$. We also know that ξ^i and η^i define affine collineations of V_n so that $\xi^i_{,j,k}+\xi^hR^i_{jkh}=\eta^i_{,j,k}+\eta^hR^i_{jkh}=0$ so that

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$$\begin{aligned}\overset{i}{\zeta}_{,j} &= \overset{h}{\xi}_{,j}\overset{i}{\eta}_{,h} - \overset{h}{\eta}_{,j}\overset{i}{\xi}_{,h} - \overset{h}{\xi}\overset{l}{\eta}R^{i}_{hjl} + \overset{h}{\eta}\overset{l}{\xi}R^{i}_{hjl} \\ &= \overset{h}{\xi}_{,j}\overset{i}{\eta}_{,h} - \overset{h}{\eta}_{,j}\overset{i}{\xi}_{,h} + \overset{h}{\xi}\overset{l}{\eta}R^{i}_{jlh}\end{aligned}$$

or

$$\zeta_{k,j} = \overset{h}{\xi}_{,j}\eta_{k,h} - \overset{h}{\eta}_{,j}\xi_{k,h} + \overset{h}{\xi}\overset{l}{\eta}R_{kjlh}.$$

Interchanging k and j and adding the two sets of equations and making use of (2.1), we find that

$$\begin{aligned}\zeta_{k,j} + \zeta_{j,k} &= g^{hl}\xi_{l,j}(\eta_{k,h} + \eta_{h,k}) + g^{hl}\xi_{l,k}(\eta_{j,h} + \eta_{h,j}) \\ &= c(\xi_{k,j} + \xi_{j,k}) = 0,\end{aligned}$$

so that ζ^i defines a motion and is therefore in G_r. Thus we have

THEOREM 2.1. *The full group G_{r+1} of homothetic mappings of V_n contains an invariant subgroup G_r of motions and a G_1 subgroup of dilations.*

Another theorem that may be proved in a similar manner is

THEOREM 2.2. *A Riemann space of constant nonzero curvature does not admit any homothetic mappings except motions.*

This is a generalization of some theorems of Sumitomo, since it is independent of compactness or orientability.

BIBLIOGRAPHY

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