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A NOTE ON COMPLETE BOOLEAN ALGEBRAS

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1. **Introduction.** Among commutative rings, Boolean algebras stand just below fields in simplicity of structure. In contrast, little is known concerning their classification. The purpose of this paper is to present a decomposition theorem for complete Boolean algebras, which, in a small way, simplifies the classification problem. As an illustration of how this decomposition theorem can be used, it is shown that an infinite cardinal number $\aleph$ can be the power of a complete Boolean algebra if and only if $\aleph^{\aleph_0}=\aleph$.

The words "complete Boolean algebra" will henceforth often be abbreviated C.B.A. Similarly, B.A. stands for Boolean algebra. The join, meet and complement operations in a B.A. are denoted $\vee$, $\wedge$ and $(')$. Inclusion is indicated by $\leqq$. Also 0 and u respectively stand for the zero and unit elements of a B.A. The least upper bound of a subset A of a C.B.A. is designated l.u.b. A.

The relation of isomorphism between Boolean algebras is denoted $\cong$. If B is a B.A. and $a\in B$, then B_a will stand for the set $\{b\in B \mid b\leqq a\}$. If $a\neq 0$, then B_a is a B.A. (which is complete if B is complete) with the join, meet and zero inherited from B and with complementation relative to a. That is, the complement of $b\leqq a$ in B_a is $b'\wedge a$. A Boolean algebra B is called homogeneous if $B_a\cong B$ for all $a\neq 0$ in B. A useful property of C.B.A.'s is they satisfy the Schroeder-Bernstein theorem: if $B\cong \overline{B}_{\bar{a}}$ and $\overline{B}\cong B_a$ ($a\in B$, $\bar{a}\in \overline{B}$), then $B\cong \overline{B}$. This fact is proved in various places, perhaps the most accessible of which is [**5**, Theorem 1.31].

The direct union of a set $\{B_\sigma \mid \sigma\in S\}$ of B.A.'s is defined in the usual way (see [**1**, p. viii]). This direct union will be denoted $\sum_{\sigma\in S} B_\sigma$. There is a useful internal characterization of direct unions of C.B.A.'s.

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Lemma 1.1. *If B is a C.B.A., then $B \cong \sum_{\sigma \in S} B_\sigma$ if and only if there exists a subset $A = \{a_\sigma | \sigma \in S\}$ of pairwise disjoint, nonzero elements of B such that* l.u.b. $A = u$ *and $B_\sigma \cong B_{a_\sigma}$ for all $\sigma \in S$.*

The proof comes directly from the definition of direct union, using the completeness of B. This lemma suggests the notation $B = \sum_{a \in A} B_a$, when A is a set of pairwise disjoint nonzero elements of B and l.u.b. $A = u$.

2. **The decomposition theorem.** The technique presented in this section is not new. For example, a particular case of it is used in Maharam's paper [**4**]. Nevertheless, it seems worthwhile to put the idea in a general setting.[1]

Definition 2.1. A *cardinal property* ν of complete Boolean algebras is a rule which assigns to each complete Boolean algebra B a unique cardinal number $\nu(B)$, such that if $B \cong \overline{B}$, then $\nu(B) = \nu(\overline{B})$.

The cardinal property is called *monotone* on B if $\nu(B_a) \leqq \nu(B_b)$ whenever $0 \neq a \leqq b$ in B.

A complete Boolean algebra B is called *ν-homogeneous* if $\nu(B_a) = \nu(B)$ for all $a \neq 0$ in B.

Of particular interest are those cardinal properties which are monotone on every C.B.A. All of the following examples enjoy this distinction.

Examples 2.2. (i) $\kappa(B)$ = cardinality of B. (ii) $\gamma(B)$ = least cardinal number of a dense subset of $B - \{0\}$. That is, $\gamma(B)$ is the least cardinal $\aleph$ for which there exists a subset A of $B - \{0\}$ with the properties $\kappa(A) = \aleph$ and for any $b \in B - \{0\}$ there exists $a \in A$ such that $a \leqq b$. (iii) $\delta(B)$ = least cardinal $\aleph$ such that if $A \subseteq B$ is a set of nonzero pairwise disjoint elements, then $\kappa(A) \leqq \aleph$.

Note that the above cardinal properties can also apply to arbitrary subsets of a B.A. (and in the case of κ, to arbitrary sets). For instance, $\delta(T)$ is the least cardinal number $\geqq$ the power of every disjointed (i.e., pairwise disjoint) subset of $T - \{0\}$.

Theorem 2.3. *Let ν be a cardinal property which is monotone on the complete Boolean algebra B. Then B decomposes uniquely in the form*

(i) $B = \sum_{a \in A} B_a$,

where

(ii) $B_a \cong \sum_{\sigma \in S_a} B_{a\sigma}$ (*not generally unique*),

with each $B_{a\sigma}$ a ν-homogeneous C.B.A. such that $\nu(B_{a\sigma}) = \aleph_a$ and $\aleph_a \neq \aleph_b$ for $a \neq b$.

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[1] This method has also been used in a more general setting by P. Erdös and A. Tarski (Ann. of Math. vol. 44 (1943) pp. 315–329).

PROOF. Let $a_{\aleph}=(\text{l.u.b. } \{b \mid \nu(B_b) \leqq \aleph\}) \wedge (\text{l.u.b. } \{b \mid \nu(B_b) < \aleph\})'$. Take A to be the set of all nonzero $a_{\aleph}$. Clearly A is a set of pairwise disjoint elements of B. If l.u.b. $A < u$, let $0 \neq c = (\text{l.u.b. } A)'$. Define $\aleph = \min \{\nu(B_b) \mid 0 \neq b \leqq c\}$. Then $0 \neq b \leqq c$ and $\nu(B_b) = \aleph$ for some b. Also, $0 \neq d \leqq b$ implies $\nu(d) \geqq \aleph$. Hence $b \leqq a_{\aleph} \leqq$ l.u.b. A, contrary to (l.u.b. $A) \wedge c = 0$. Thus, l.u.b. $A = u$. By 1.1, $B = \sum_{a \in A} B_a$.

For each $\aleph$ such that $a_{\aleph} \neq 0$, let $A_{\aleph}$ be a maximal set of nonzero, pairwise disjoint elements $b \leqq a_{\aleph}$ satisfying $\nu(B_b) = \aleph$. Then

$$\text{l.u.b. } A_{\aleph} = a_{\aleph}.$$

Indeed, if $0 \neq c = a_{\aleph} \wedge (\text{l.u.b. } A_{\aleph})'$, then $d = c \wedge b \neq 0$ for some b satisfying $\nu(B_b) \leqq \aleph$. Hence, by monotonicity, $\nu(B_d) \leqq \nu(B_b) \leqq \aleph$. On the other hand, $d \leqq a_{\aleph} \leqq (\text{l.u.b.}\{b \mid \nu(B_b) < \aleph\})'$ implies $\nu(B_d) \geqq \aleph$. Thus, $\nu(B_d) = \aleph$ and the maximality of $A_{\aleph}$ is contradicted. This proves $B_{a_{\aleph}} = \sum_{b \in A_{\aleph}} B_b$. Moreover, each B_b $(b \in A_{\aleph})$ is ν-homogeneous with $\nu(B_b) = \aleph$, since, as was just noted, $0 \neq d \leqq b \in A_{\aleph}$ implies $\aleph \leqq \nu(B_d) \leqq \nu(B_b) \leqq \aleph$.

It remains to observe that the decomposition (i) is unique. This is clear since, subject to condition (ii), the unit a of B_a is the join of elements b such that $\nu(B_b) = \aleph_a$. On the other hand, if $0 \neq c \leqq a$, then $b \wedge c \neq 0$ for some b such that B_b is ν-homogeneous with $\nu(B_b) = \aleph_a$. Therefore $\aleph_a = \nu(B_{b \wedge c}) \leqq \nu(B_c)$. Consequently $a = a_{\aleph_a}$.

3. **The power of complete Boolean algebras.** Professor B. Jónsson has proposed the following question: what are the possible cardinalities of complete, homogeneous Boolean algebras? In this section Jónsson's question will be answered.[2] Moreover, by using 2.3, it will be possible to determine the cardinalities of arbitrary complete Boolean algebras.

First note that any finite B.A. is complete. It is well known (see [**1**, p. 159]) that any finite B.A. has 2^n elements for some integer n. Thus, only infinite C.B.A.'s are to be considered.

THEOREM 3.1. *If B is an infinite C.B.A. of cardinality $\aleph$, then* $\aleph^{\aleph_0} = \aleph$.

PROOF. By 2.3, $B \cong \sum_{\sigma \in S} B_\sigma$, where each B_σ is κ-homogeneous. In particular, if B_σ is finite, then $\kappa(B_\sigma) = 2$. Let $S = S' \cup S''$, where S' consists of all σ such that $\kappa(B_\sigma) = 2$ and S'' are those σ for which B_σ is infinite. Then clearly $\aleph = \kappa(B) = 2^{\kappa(S')} \cdot \prod_{\sigma \in S''} \kappa(B_\sigma)$. If $\kappa(S')$ is finite, then (since $\aleph$ is infinite) $2^{\kappa(S')} \cdot \prod_{\sigma \in S''} \kappa(B_\sigma) = \prod_{\sigma \in S''} \kappa(B_o)$

[2] Seymour Ginsburg (Proc. Amer. Math. Soc. vol. 9 (1958) pp. 130–132) has shown that complete homogeneous Boolean algebras of power $2^{\aleph}$ exist for any infinite cardinal $\aleph$.

and $\aleph^{\aleph_0} = \prod_{\sigma \in S''} \kappa(B_\sigma)^{\aleph_0}$. If $\kappa(S')$ is infinite, $\aleph^{\aleph_0} = 2^{\aleph_0 \cdot \kappa(S')} \cdot \prod_{\sigma \in S''} \kappa(B_\sigma)^{\aleph_0} = 2^{\kappa(S')} \cdot \prod_{\sigma \in S''} \kappa(B_\sigma)^{\aleph_0}$. In either case, it is sufficient to prove: if B_σ is an infinite, κ-homogeneous C.B.A., then $\kappa(B_\sigma)^{\aleph_0} = \kappa(B_\sigma)$.

If B_σ is an infinite C.B.A., then clearly $\delta(B_\sigma) \geqq \aleph_0$. Hence $B_\sigma = \sum_{i=1}^{\infty} B_i$, where $B_i = B_{a_i}$ for some $a_i \neq 0$ in B. If B_σ is also κ-homogeneous, then $\kappa(B_\sigma) = \kappa(B_i)$ for all i. Hence, $\kappa(B_\sigma) = \prod_{i=1}^{\infty} \kappa(B_i) = \kappa(B_\sigma)^{\aleph_0}$. This proves the theorem. The following lemma, relating the cardinal properties κ, γ and δ of 2.2, prepares the way for the converse of 3.1.

Lemma 3.2. *Let B be a B.A. and T a dense subset of B such that* (i) $\kappa(T) = \aleph_\alpha$, (ii) $\delta(T) = \aleph_\beta$, (iii) $(\aleph_\alpha)^{\aleph_\beta} = \aleph_\alpha$. *Then* $\kappa(B) = \aleph_\alpha$.

Proof. Since T is dense in B, every nonzero element of B is the least upper bound of some disjointed subset of T. Indeed, if $b \neq 0$ in B and X is a maximal disjoined subset of T with $x \leqq b$ for all $x \in X$, then l.u.b. $X = b$. For otherwise, there exists $c \neq 0$ in B with $c \leqq b$ and $c \wedge x = 0$ for all $x \in X$. Choosing $t \in T$ such that $0 \neq t \leqq c$ gives $X \cup \{t\}$—a disjointed subset of T composed of elements $\leqq b$. Since $c \wedge x = 0$ for all $x \in X$, it cannot happen that $t \in X$. Therefore $X \cup \{t\}$ properly contains X. But this contradicts the maximality of X. Thus, l.u.b. $X = b$, as claimed. Since $\delta(T) = \aleph_\beta$, it follows that every element of B is a join of some subset X of T with $\kappa(X) \leqq \aleph_\beta$. Thus $\aleph_\alpha = \kappa(T) \leqq \kappa(B) \leqq (\aleph_\alpha)^{\aleph_\beta} = \aleph_\alpha$.

Theorem 3.3. *If $\aleph$ is an infinite cardinal number satisfying $\aleph^{\aleph_0} = \aleph$, then there is a complete, homogeneous Boolean algebra of power $\aleph$.*

Proof. Let S be a set of cardinality $\aleph$. For $\sigma \in S$, denote $X_\sigma = \{x_{\sigma 1}, x_{\sigma 2}\}$, the two element discrete space. Let $X = \prod_{\sigma \in S} X_\sigma$ be the cartesian product space. Then X is a totally disconnected, compact Hausdorff space. Let B be the C.B.A. of regular open sets of X (see [1, p. 177]). Let T be all subsets of X which are of the form:

$$Y = \bigcap_{\sigma \in F} \overset{-1}{\pi_\sigma}(\{x_{\sigma f(\sigma)}\}),$$

where F is a finite subset of X, $f \in \{1, 2\}^F$ and $\pi_\sigma: X \to X_\sigma$ is the usual component projection. These sets are clearly open-and-closed (hence regular open) and form a basis for the topology of X. Thus, T is a dense subset of B. Also, as a topological space, each Y is a cartesian product of $\aleph$ 2-point spaces, thus homeomorphic to X. It follows that $B \cong B_Y$ for all such Y. Since T is dense, the Schroeder-Bernstein theorem implies that B is homogeneous. It is clear that $\kappa(T) = \aleph + \aleph^2 + \cdots = \aleph$. Hence, according to 3.2, the proof is completed by the observation $\delta(T) = \aleph_0$. The proof of this fact is sketched in [**3**, p. 166].

REMARK. The C.B.A. constructed in 3.3 is precisely the completion by cuts of the free B.A. with $\aleph$ generators.

Combining Theorems 3.1 and 3.3 wins the objective of this section: an infinite cardinal $\aleph$ can be the power of a complete, or complete homogeneous Boolean algebra if and only if $\aleph^{\aleph_0} = \aleph$. One may ask what cardinals satisfy $\aleph^{\aleph_0} = \aleph$. This equality prevails if $\aleph$ is of the form $2^{\aleph'}$ for some infinite $\aleph'$. It does not hold if $\aleph$ is a countable sum of smaller cardinal numbers (by König's theorem—see [**2**, p. 34]). Little more can be said with certainty. But if the generalized continuum hypothesis is assumed, the situation becomes clearer: $(\aleph_\alpha)^{\aleph_0} = \aleph_\alpha$ if α is not a limit ordinal, or if α is a limit ordinal which (considered as a well ordered set) contains no countable, cofinal subset (see [**6**, p. 9, Theorem 7]); otherwise $(\aleph_\alpha)^{\aleph_0} > \aleph_\alpha$.

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