COMPARISON OF SUBTHEORIES[1]

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Let ZF denote the Zermelo-Fraenkel set theory and ZFI the theory obtained from it by the addition of an axiom which asserts the existence of at least one inaccessible number. If ZF is consistent then ZFI contains number-theoretic theorems which are not theorems of ZF, e.g. the arithmetical sentence which asserts the consistency of ZF—Con(ZF). Mostowski [**4**] introduced a sentence of the theory of real numbers which can be proved in ZFI but cannot be proved in ZF. Let S_1 and S_2 be two theories such that the relation of S_1 to S_2 is like that of ZFI to ZF. The present paper will show to what extent the subtheories of S_1 contain more theorems than the corresponding subtheories of S_2.

We shall use the terminology of Tarski-Mostowski-Robinson [**8**].[2] A set of symbols of the first order predicate calculus with the usual formation rules for terms and formulae, as laid out in [**8**, pp. 6–7], will be called a standard formal language. Each theory with standard formalization is formalized in a standard formal language. If R is a standard formal language and P an unary predicate (P need not be a symbol of R) we construct a new formal language $R^{(P)}$ by the relativization of quantifiers in R to P (see [**8**, p. 24]). If A is a sentence of R we denote by $A^{(P)}$ the sentence obtained from A by relativizing the quantifiers in it to P. Let Q be a theory with standard formalization. We define the notion of an interpretation of R in Q as in [**8**, pp. 20–21, 29]. This interpretation is obtained by a new theory with standard formalization $Q(R)$ which is formalized in the language which consists of the symbols of both Q and $R^{(P)}$. The valid sentences of $Q(R)$ are exactly those which are derivable from the set which consists of the valid sentences of Q and possible definitions of the nonlogical constants of $R^{(P)}$ in Q. Given a particular interpretation of R in Q we can define in the language R a theory with standard formalization Q/R (which may be called the theory induced by Q in R) by the following stipulation: A sentence A of R is valid in Q/R

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[2] Except for the word "subtheory" which has here a sense somewhat different from that in [**8**].

if and only if $A^{(P)}$ is valid in $Q(R)$. Let T be a given theory with standard formalization with a given interpretation in Q. We can again define the theory Q/T. We observe that Q/T is an extension of T without new symbols.

Let Q and T be theories with standard formalization. Q is said to be an essentially infinite extension of T if Q is a consistent extension of T and no consistent extension of Q without new symbols is a finite extension of T.[3]

THEOREM. *Let S_1 and S_2 be consistent theories with standard formalization and let S_2 be an axiomatic theory. Let R be a standard formal language. Let ϕ_2 be a given interpretation of R in S_2. Denote $S_2(R)$ by S_2'. Let ϕ_1 be a given interpretation of S_2' in S_1. Denote $S_1(S_2')$ by S_1'. A^* will denote*[4] *$A^{(P_2)}$ if A is a sentence of R and $A^{(P_1)}$ if A is a sentence of S_2'. Let S_2/R contain number-theory. A_n will denote the formula of S_2' whose Gödel-number is n. Let there exist a truth definition*[5] *$T(n)$ for S_2' in S_1' with the following properties:*

(a) *For each*[6] *$\mathbf{n}$ which is a Gödel-number of a sentence of*

$$S_2' \; S_1' \vdash T(\mathbf{n}) \equiv A_n^*.$$

(b) *If $Ax_{s_2'}(n)$ is the recursive number-theoretic formula which asserts that n is the Gödel-number of an axiom of S_2' then $S_1' \vdash (n)(Ax_{s_2'}(n) \supset T(n))$.*

(c) *Let $C(n)$ be the primitive recusive function which maps the Gödel-numbers of formulae of S_2' on the Gödel-numbers of their universal closures. Denote $U(n) \equiv T(C(n))$. If A_n follows in S_2' from A_l and A_m by one of the rules of inference then $U(l) \wedge U(m) \supset U(n)$.*

(d) *If $\mathbf{l}$ is the Gödel-number of $O=1$ then $S_1' \vdash \sim T(\mathbf{l})$.*

(e) *$S_1' \vdash W(0) \wedge (n)(W(n) \supset W(n+1)) \supset (n)W(n)$ where $W(n)$ is any formula of S_1' which is obtained from number theoretic formulae of S_1' and $T(n)$ by the propositional connectives and quantification of number variables.*

Under these conditions S_1/R is[7] *an essentially infinite extension of S_2/R.*

PROOF. Let Q be a consistent extension without new symbols of S_1/R. We have to prove that Q is an infinite extension of S_2/R.

[3] This concept is introduced in analogy to Montague [**3**].

[4] P_1 and P_2 are the unary predicates used for the relativization of the quantifiers for ϕ_1 and ϕ_2 respectively.

[5] The natural numbers of S_1' are the interpretation in S_1' of the interpretation in S_2' of the natural numbers in S_2/R.

[6] Boldface letters denote numerals.

[7] By S_1/R we mean $(S_1/S_2)/R$.

Assume that Q is a finite extension of S_2/R, i.e., Q can be obtained from S_2/R by the addition of a single axiom B. Let S_1'' (resp. S_2'') denote the theory which is obtained from S_1' (resp. S_2') by the addition of the axiom B^{**} (resp. B^*). Let $\boldsymbol{m}$ be the Gödel-number of B^*. By (a) we have $S_1' \vdash B^{**} \equiv A^*_{\boldsymbol{m}} \equiv T(\boldsymbol{m})$, and hence $S_1'' \vdash T(\boldsymbol{m})$. The axioms of S_2'' are those of S_2' with the addition of

$$B^* - Ax_{S_2''}(n) \equiv Ax_{S_2'}(n) \vee n = \boldsymbol{m}.$$

By (b) and $T(\boldsymbol{m})$ we have $S_1'' \vdash (n)(Ax_{S_2''}(n) \supset T(n))$. Let $Th_{S_2''}(n)$ be the number-theoretic formula which asserts that n is the Gödel-number of a theorem of S_2''. Let A_n be a theorem of S_2'' and $A_{n_1}, \cdots, A_{n_j}$ $(n_j = n)$ a proof-sequence of A_n. $U(n_i)$ can be easily seen to be obtained from $T(n)$ and number-theoretic formulae by the propositional connectives and quantification of number variables. Since we have $S_1'' \vdash (n)(Ax_{S_2''}(n) \supset T(n))$ and (c), (e) enables us to prove $U(n_i)$, $1 \leqq i \leqq j$, by induction on i and thus we obtain $S_1'' \vdash U(n)$, and since A_n is a sentence we have $S_1'' \vdash T(n)$. Thus we proved $S_1'' \vdash (n)(Th_{S_2''}(n) \supset T(n))$. Therefore we have by (d) $S_1'' \vdash \mathrm{Con}(S_2'')^{**}$, and hence, by the deduction theorem, $S_1' \vdash B^{**} \supset \mathrm{Con}(S_2'')^{**}$, $S_1/R \vdash B \supset \mathrm{Con}(S_2'')$. Q, which is obtained from S_2/R by the addition of B, is an extension of S_1/R and hence $Q \vdash B \supset \mathrm{Con}(S_2'')$, $S_2/R \vdash B \supset (B \supset \mathrm{Con}(S_2''))$, i.e., $S_2/R \vdash B \supset \mathrm{Con}(S_2'')$ and therefore $S_2' \vdash B^* \supset \mathrm{Con}(S_2'')^*$. Since B^* is an axiom of S_2'' we have $S_2'' \vdash \mathrm{Con}(S_2'')^*$. But $\mathrm{Con}(S_2'')^*$ is the arithmetical sentence of S_2'' asserting the consistency of S_2'', and hence, by Gödel's theorem, S_2'' is inconsistent, i.e., $\sim B^*$ is a theorem of S_2'. Therefore, $\sim B$ is a theorem of S_2/R and a fortiori $\sim B$ can be proved in S_1/R; thus contradicting the assumption that Q is a consistent extension of S_1/R.

If R is the langue of number-theory, S_1—set-theory with the axiom of substitution and S_2—set-theory without that axiom, ϕ_1 the interpretation defined by the model of S_2 in S_1 of Bernays [1] and ϕ_2 the usual interpretation of number theory in set theory then by the results of Mendelson [2] (in which the axiom $V = L$ can be omitted) there is a truth definition which fulfills the conditions of our theorem. Thus we prove that the number theory of set theory with the axiom of substitution is an essentially infinite extension of the number theory of set theory without that axiom.[8] In the same way we obtain

[8] In this case $S_1/R(=(S_1/S_2)/R)$, which is obviously the number theory of the model of S_2 in S_1, is really what we call the number theory of S_1. This follows from the result that the notion of the natural number is absolute with respect to interpretations like ϕ_1 (see for example Shepherdson [6, 2.322]). The same reasoning is also valid for the other examples which will be mentioned.

the result that the number theory (and also other subtheories, e.g., the theory of real numbers) of set theory with an axiom which asserts the existence of an inaccessible number is an essentially infinite extension of the corresponding subtheory of set theory without such an axiom, etc.

Let R and S_2 be the set theory of Zermelo-Fraenkel, let S_1 be the set theory of Bernays-Gödel and let ϕ_1 and ϕ_2 be the usual interpretations. Mostowski [5] pointed out that there is a truth definition $T(n)$ for S_2 in S_1 but for which conditions (b) and (e) of our theorem are not fulfilled. The result of our theorem does not hold since Mostowski [5] remarks that S_1/R coincides with S_2.[9] If we add to S_1 an induction axiom-schema $A(0)\wedge(n)(A(n)\supset A(n+1))\supset(n)A(n)$, where A is any formula of S_1 (in which bound class variables may also occur) the situation is entirely changed since, as follows easily from Mostowski [5], conditions (b) and (e) are also fulfilled with respect to Mostowski's truth definition and by our theorem S_1/R is an essentially infinite extension of S_2.

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[9] See also Shoenfield [7].