THE SIMPLICITY OF CERTAIN NONASSOCIATIVE ALGEBRAS[1]

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Introduction. Let V be an additively written elementary p-group of order p^n, where $n>1$, and let $\phi(u, v)$, $\psi(u, v)$ be two alternate bi-additive functions defined on V and with values in a field K of characteristic p. Let k be a fixed nonzero element in V, and assume that the following two conditions are satisfied:

(0.1) If α, β are nonzero elements in K then $\alpha\phi(u, v)+\beta\psi(u, v)=0$ for all v implies $u=0$;

(0.2) $\phi(k, v)=0$ for all v.

If ϕ is identically zero, then the conditions (0.1)–(0.2) are equivalent to saying that ψ is nondegenerate. Define an algebra L over K with basis

$$\{e_u \mid u \in V, u \neq 0, u \neq k\}$$

by the multiplication table:

$$e_u e_v = \phi(u, v)e_{u+v} + \psi(u, v)e_{u+v+k},$$

where we set $e_0=e_k=0$. The purpose of this note is to show that the algebra L is simple if $p>2$.

An elementary computation shows that

$$(e_u e_v)e_w + (e_v e_w)e_u + (e_w e_u)e_v$$
$$= (\psi(u, v)\psi(k, w) + \psi(v, w)\psi(k, u) + \psi(w, u)\psi(k, v))e_{u+v+w+2k}.$$

From this we may deduce easily that L is a Lie algebra if and only if there exist two additive functions f and g on V with values in K such that

$$(0.3) \qquad \psi(u, v) = f(u)g(v) - f(v)g(u)$$

for all u, v in V. In the case where $\psi(u, v)$ is given as in (3), therefore, the algebra L is a special case of the algebras considered by Richard Block [2] if the following conditions are satisfied: V is the direct sum of subgroups V_1 and V_2 such that $k\in V_2$; $\phi(u, v)=0$ for all $u\in V_2$, $v\in V$; $\psi(u, v)=0$ for all $u\in V_1$, $v\in V$. If $\psi(u, v)$ cannot be expressed as in (3), then L is of course not a Lie algebra. But it gives an interesting family of simple nonassociative algebras.

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1. **The center of** L. As a first step in proving the simplicity of L we show that the center of L is zero. Since V is an elementary p-group, we may express V as the direct sum of a subgroup U and the subgroup generated by k. Then any element x in L can be written in the form

$$x = \sum_{u \in U} \sum_{i=0}^{p-1} \alpha_{ui} e_{u+ik} \tag{1.1}$$

with $\alpha_{u,i}$ in K, where $\alpha_{00} = \alpha_{01} = 0$. For the element x given by (1.1) define x_u for each u in U by

$$x_u = \sum_{i=0}^{p-1} \alpha_{u,i} e_{u+ik}.$$

Then $x = \sum x_u$, and if x is in the center of L then for each u in U the element x_u is also in the center of L. Therefore if the center of L is not zero, then there must be in the center a nonzero element x of the form

$$x = \alpha_0 e_u + \alpha_1 e_{u+k} + \cdots + \alpha_{p-1} e_{u+(p+1)k}, \tag{1.2}$$

where $\alpha_i \in K$. For the element x given by (1.2) define the *length* $\lambda(x)$ of x to be the number of nonzero coefficients α_i. Let the above x be of minimal length among the nonzero central elements such that $x = x_u$ for some u. We consider the following two cases:

CASE I. $u = 0$. Then x is of the form

$$x = \alpha_2 e_{2k} + \alpha_3 e_{3k} + \cdots + \alpha_{p-1} e_{(p-1)k}.$$

Take an element $v \in U$ such that $\psi(k, v) \neq 0$. Then

$$xe_v = \alpha_2 \psi(2k, v) e_{3k+v} + \cdots + \alpha_{p-1} \psi((p-1)k, v) e_v = 0.$$

Hence $\alpha_2 = \cdots = \alpha_{p-1} = 0$; $x = 0$, a contradiction.

CASE II. $u \neq 0$. We have

$$xe_{2k} = \sum_{i=0}^{p-1} \alpha_i \psi(u + ik, 2k) e_{u+(i+2)k} = 0.$$

Since $u + (i+2)k \neq 0$, k for all i, and since some $\alpha_i \neq 0$, we have $\psi(k, u) = 0$. Let $v \in V$ be such that $\psi(k, v) \neq 0$. Then for every i, $u + v + ik$ is neither zero nor k, since $\psi(k, u+v+ik) \neq 0$. Now we have

$$xe_v = \sum_{i=0}^{p-1} (\alpha_i \phi(u, v) + \alpha_{i-1} \psi(u + (i-1)k, v)) e_{u+v+ik},$$

where $\alpha_{-1} = \alpha_{p-1}$. Since $xe_v = 0$, we have

$$\alpha_i\phi(u, v) + \alpha_{i-1}\psi(u + (i-1)k, v) = 0 \tag{1.3}$$

for all i. If w is an element such that $\psi(k, w)=0$ then $\psi(k, v+w)\neq 0$. Hence we may replace v in (1.3) by $v+w$. Then it follows that (1.3) is true for all $v\in V$ and all i. Let $i=1$ in (1.3) and use (0.1). Then we have $\alpha_0=0$ or $\alpha_1=0$. Take an i such that $\alpha_i\neq 0$, $\alpha_{i-1}=0$. Then from (1.3) it follows that

$$\phi(u, v) = 0 \tag{1.4}$$

for all v in V. Let j be such that $\alpha_j=0$, $\alpha_{j-1}\neq 0$. Then from (1.3) we have

$$\psi(u + (j-1)k, v) = 0 \tag{1.5}$$

for all v in V. Also from (1.4) we have $\phi(u+(j-1)k, v)=0$ for all $v\in V$. Hence we have $u+(j-1)k=0$, a contradiction.

2. **Minimal elements.** For x in L denote by $\rho(x)$ the number of nonzero x_u, where $u\in U$. Let M be a fixed nonzero ideal of L, and call a nonzero element x in M *minimal* if $\rho(x)$ is minimal. First we prove

LEMMA 2.1. *There exists a minimal element x in M for which $x_0\neq 0$.*

PROOF. We shall derive a contradiction by assuming that the lemma is not true. Let $x=\sum x_u$ be a minimal element. If there exists an element u in U such that $x_u\neq 0$ and $\psi(k, u)\neq 0$, then for any integer j the element $u+jk$ is neither 0 nor $-k$. Hence we may consider the product $y=e_{-u-jk}x$. It is readily seen that $\psi(k, u)\neq 0$ implies that $y_0\neq 0$ for some j. Clearly $\rho(y)\leqq\rho(x)$ and $y\in M$. Hence it follows from our assumption that, for any minimal element x, $x_u\neq 0$ implies $\psi(k, u)=0$. Take an element v in V such that $\psi(k, v)\neq 0$. Then $x_ue_v=0$ for any minimal element x, because for $y=xe_v$ we have $\rho(y)\leqq\rho(x)$ and if $y_w\neq 0$ then $w=u+v$, where $\psi(k, u)=0$, and hence $\psi(k, w)\neq 0$ which contradicts the conclusion obtained above. Now consider elements v' in V such that $\psi(k, v')=0$, $v'\neq 0$, $v'\neq k$. Then $\psi(k, v+v')\neq 0$, and hence $x_ue_{v+v'}=0$. Let

$$x_u = \alpha_0e_u + \alpha_1e_{u+k} + \cdots + \alpha_{p-1}e_{u+(p-1)k}. \tag{2.1}$$

Then $x_ue_v=0$ implies

$$\alpha_i\phi(u, v) + \alpha_{i-1}\psi(u + (i-1)k, v) = 0$$

for all i, where we set $\alpha_{-1}=\alpha_{p-1}$, because $u+v+ik$ is neither zero nor k. Similarly we have

$$\alpha_i\phi(u, v + v') + \alpha_{i-1}\psi(u + (i-1)k, v + v') = 0$$

and hence from (2.1) we have

$$\alpha_i\phi(u, v') + \alpha_{i-1}\psi(u + (i - 1)k, v') = 0$$

for all i, which shows that $x_u e_{v'} = 0$. Thus we have shown that x belongs to the center of L. By the result in the first section, $x = 0$. This is a contradiction. Thus Lemma 2.1 is proved.

Let x be a minimal element in the ideal M such that $x_0 \neq 0$ and such that the number of nonzero terms in x_0 is as small as possible. Let

$$x_0 = \alpha_2 e_{2k} + \alpha_3 e_{3k} + \cdots + \alpha_{p-1} e_{(p-1)k}.$$

Take $v \in V$ such that $\psi(k, v) \neq 0$, and consider $y = (xe_{v-k})e_{-v-k}$. It is clear that either $y = 0$ or y is a minimal element in M, and that $y_0 = (x_0 e_{v-k})e_{-v-k}$. We have

$$y_0 = -\psi(k, v)^2 \sum_{i=2}^{p-1} (i - 1)i\alpha_i e_{ik}.$$

Let j, $2 \leqq j \leqq p-1$, be such that $\alpha_2 = \cdots = \alpha_{j-1} = 0$, $\alpha_j \neq 0$, and consider $z = -\psi(k, v)^2(j-1)jx - y$. It is clear that $\rho(z) \leqq \rho(x)$, and that the number of nonzero terms in z_0 is smaller than that of x_0. Hence by the assumption on x we have $z = 0$. Then for any t such that $\alpha_t \neq 0$, we have $j(j-1) \equiv t(t-1) \pmod{p}$. Hence $t = j$ or $t = p - j + 1$. Therefore x_0 consists of only one nonzero term or is of the form

$$x_0 = \alpha e_{jk} + \beta e_{(p-j+1)k},$$

where α, β are nonzero elements in K. We rule out the second possibility as follows: We may assume without loss of generality that $2 \leqq j < (p+1)/2$. Let again v be such that $\psi(k, v) \neq 0$, and consider $y' = (xe_v)e_{-v+(j-2)k}$. Clearly $\rho(y') \leqq \rho(x)$ and $y_0' = (x_0 e_v)e_{-v+(j-2)k}$. We have

$$x_0 e_v = \psi(k, v)(j\alpha e_{v+(j+1)k} + (1 - j)\beta e_{v+(2-j)k}).$$

Since $e_w e_{-w} = 0$ for any w, we have from the above

$$y_0' = -\psi(k, v)^2 j(2j - 1)\alpha e_{2jk} \neq 0.$$

Thus we have proved that x_0 consists of a single nonzero term. Let $x = e_{jk} + \sum_{u \neq 0} x_u$, where $2 \leqq j < p - 1$. Take $v \in V$ such that $\psi(k, v) \neq 0$, and consider $x' = (xe_v)e_{-v-k}$. We have

$$x' = -j^2\psi(k, v)^2 e_{(j-1)k} + \sum (x_u e_v)e_{-v-k}.$$

Hence we may assume that $j = p - 1$; $x = e_{-k} + \sum_{u \neq 0} x_u$. Then from the above we have $x_0' = 0$. Hence $\rho(x') < \rho(x)$; $x' = 0$; $(x_u e_v)e_{-v-k} = 0$ for all nonzero u in U. Similarly $(x_u e_v)e_{-v} = 0$ for all $u \in U$. Let

$$x_u e_v = \sum_{i=0}^{p-1} \beta_i e_{u+v+ik}.$$

Then $(x_u e_v)e_{-v} = 0$ implies

$$\beta_i \phi(u, -v) + \beta_{i-1}\psi(u + (i-1)k, -v) = 0$$

for all i; $(x_u e_v)e_{-v-k} = 0$ implies

$$\beta_i \phi(u, -v) + \beta_{i-1}\psi(u + (i-2)k, -v - k) = 0$$

for all i. Hence it follows that $\beta_{i-1}\psi(u+v, k) = 0$ for all i whenever $\psi(v, k) \neq 0$. Using $2v$ instead of v we have $\beta_{i-1}\psi(u+2v, k) = 0$. Therefore $\beta_i = 0$ for all i (note that $\beta_{-1} = \beta_{p-1}$). Thus we have shown that $x_u e_v = 0$ whenever $\psi(k, v) \neq 0$. Then by proceeding as in the proof of Lemma 2.1 we may conclude that $x_u = 0$ for all nonzero u in U. Thus it is shown that the ideal M contains e_{-k}.

3. **The simplicity of** L. Let M be an arbitrary nonzero ideal of L. By the preceding section M contains e_{-k}, and since $e_{-k}e_v = -\psi(k, v)e_v$, M also contains e_v for every v such that $\psi(k, v) \neq 0$. Consider now an element u in V such that $\psi(k, u) = 0$, $u \neq 0$, $u \neq k$, $u \neq -k$.

CASE I. $\psi(u, w) = 0$ for all w in V. In this case take an element v in V such that $\psi(k, v) \neq 0$. Since e_{-v} is in M, $e_{u+v}e_{-v} = \phi(u, -v)e_u$ is in M. Hence if e_u is not in M, then $\phi(u, v) = 0$ for all v such that $\psi(k, v) \neq 0$. Let $\psi(k, v') = 0$. Then $\psi(k, v+v') \neq 0$; $\phi(u, v+v') = 0$; $\phi(u, v') = 0$. Thus $\psi(u, w) = \phi(u, w) = 0$ for all w in V. Then by (0.1) we have $u = 0$, a contradiction. In this case, therefore, e_u is contained in M.

CASE II. $\psi(u, w) \neq 0$ for some w in V. In this case there exists $v \in V$ such that $\psi(u, v) \neq 0$, $\psi(k, v) \neq 0$, for if $\psi(u, v) = 0$ for all v such that $\psi(k, v) \neq 0$, then it may be proved as in Case I that $\psi(u, w) = 0$ for all w in V. Take an arbitrary v such that $\psi(k, v) \neq 0$. Then, since e_{-v-k} is in M, $e_{u+v}e_{-v-k}$ is also in M. We have

$$e_{u+v}e_{-v-k} = \phi(u, -v)e_{u-k} + \psi(u, -v)e_u \in M. \tag{3.1}$$

If $u = 2k$ then from (3.1) it follows that e_u is in M, since there exists v such that $\psi(u, v) \neq 0$, $\psi(k, v) \neq 0$. Therefore we may assume $u \neq 2k$. If e_u is not in M then from (3.1) it follows that there exist $\alpha \neq 0$ and β in K such that

$$\alpha\phi(u, v) + \beta\psi(u, v) = 0 \tag{3.2}$$

for all v for which $\psi(k, v) \neq 0$. If $\psi(k, v') = 0$ then $\psi(k, v+v') \neq 0$, and hence we may replace v in (3.2) by $v+v'$. From this we may infer that (3.2) is true for all $v \in V$. Since $u \neq 0$, (0.1) now implies that $\beta = 0$. Then from (3.2) it follows that $e_u \in M$, a contradiction. Thus we have shown that every basis element is contained in M; $M=L$. Thus the simplicity of L is proved.

4. **Remarks.** The algebra L considered above may be generalized in the obvious way if we use $m+1$ biadditive functions $\phi_0, \phi_1, \cdots, \phi_m$ instead of ϕ and ψ, and m constant elements $k_1, \cdots, k_m$ instead of k (see [2] for instance). The author, however, has been unable to obtain any result in this direction. Finally we remark that the Lie algebra L and its generalizations mentioned above are all obtained as special cases of Lie algebras constructed as follows: Let $\mathfrak{A}$ be the group algebra over K of an elementary p-group, and let $(D_1, \cdots, D_r)$ be an orthogonal system (see [3] for definition) of derivations of $\mathfrak{A}$. Let $a_1, \cdots, a_r$ in $\mathfrak{A}$ be such that $D_i a_j = D_j a_i$ for all i, j. Suppose that r^2 elements a_{ij}, $(i, j=1, \cdots, r)$, in $\mathfrak{A}$ satisfy the following conditions:

$$(4.1) \qquad a_{ij} + a_{ji} = 0; \qquad a_{ii} = 0$$

for all i and j;

$$(4.1) \qquad \sum_{s=1}^{r} (a_{is} D_s a_{jk} + a_{js} D_s a_{ki} + a_{ks} D_s a_{ij} + a_{is} a_s a_{jk} + a_{js} a_s a_{ki} + a_{ks} a_s a_{ij}) = 0,$$

for all i, j, and k.

Then an elementary computation shows that the set L of derivations of $\mathfrak{A}$ of the form

$$E_f = \sum_{s,t} a_{st}(D_t f - a_t f) D_s$$

where f runs over $\mathfrak{A}$, forms a subalgebra of the derivation algebra of $\mathfrak{A}$. We have $[E_f, E_g] = E_h$, where

$$h = \sum_{s,t} a_{st}(D_t f - a_t f)(D_s g - a_s g).$$

The Lie algebras considered in this note and those in [2] can be obtained by taking some special values of a_i, a_{ij}. In case det (a_{ij}) is a unit in $\mathfrak{A}$, we may simplify the conditions (4.1)–(4.2) by using $(b_{ij}) = (a_{ij})^{-1}$. It may be seen easily that (4.1)–(4.2) are equivalent to the following:

$$b_{ij} + b_{ji} = 0; \qquad b_{ii} = 0;$$

$$(D_i b_{jk} - a_i b_{jk}) + (D_j b_{ki} - a_j b_{ki}) + (D_k b_{ij} - a_k b_{ij}) = 0$$

for all i, j, and k.

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A NOTE ON COMPLETE BOOLEAN ALGEBRAS

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1. **Introduction.** Among commutative rings, Boolean algebras stand just below fields in simplicity of structure. In contrast, little is known concerning their classification. The purpose of this paper is to present a decomposition theorem for complete Boolean algebras, which, in a small way, simplifies the classification problem. As an illustration of how this decomposition theorem can be used, it is shown that an infinite cardinal number $\aleph$ can be the power of a complete Boolean algebra if and only if $\aleph^{\aleph_0}=\aleph$.

The words "complete Boolean algebra" will henceforth often be abbreviated C.B.A. Similarly, B.A. stands for Boolean algebra. The join, meet and complement operations in a B.A. are denoted $\vee$, $\wedge$ and ($'$). Inclusion is indicated by $\leqq$. Also 0 and u respectively stand for the zero and unit elements of a B.A. The least upper bound of a subset A of a C.B.A. is designated l.u.b. A.

The relation of isomorphism between Boolean algebras is denoted $\cong$. If B is a B.A. and $a\in B$, then B_a will stand for the set $\{b\in B \mid b\leqq a\}$. If $a\neq 0$, then B_a is a B.A. (which is complete if B is complete) with the join, meet and zero inherited from B and with complementation relative to a. That is, the complement of $b\leqq a$ in B_a is $b'\wedge a$. A Boolean algebra B is called homogeneous if $B_a\cong B$ for all $a\neq 0$ in B. A useful property of C.B.A.'s is they satisfy the Schroeder-Bernstein theorem: if $B\cong \bar{B}_{\bar{a}}$ and $\bar{B}\cong B_a$ ($a\in B$, $\bar{a}\in \bar{B}$), then $B\cong\bar{B}$. This fact is proved in various places, perhaps the most accessible of which is [**5**, Theorem 1.31].

The direct union of a set $\{B_\sigma \mid \sigma\in S\}$ of B.A.'s is defined in the usual way (see [**1**, p. viii]). This direct union will be denoted $\sum_{\sigma\in S} B_\sigma$. There is a useful internal characterization of direct unions of C.B.A.'s.

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