ON RINGS OF (γ, δ)-TYPE

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1. **Introduction.** Algebras of (γ, δ) type have been studied in [1] and [2] and the structure of algebras with $\delta \neq 0, 1$ was studied in [2]. In this paper we consider the cases $\delta = 0, 1$ but we shall enlarge our scope to consider rings. When $\delta = 0$, the ring is left alternative or $(\gamma, \delta) = (-1, 0)$. If $\delta = 1$, the ring is right alternative or $(\gamma, \delta) = (1, 1)$. The pairs $(-1, 0)$ and $(1, 1)$ are perhaps the most important since they are residual cases in the classification of almost alternative algebras relative to quasiequivalence [1, Theorem 6]. The main result is that a simple ring whose characteristic is not 2 and which contains an idempotent element e is either associative or e is the unity element of the ring. A ring R is said to be of (γ, δ) type if it satisfies the identities

$$(z, x, y) + \gamma(x, z, y) + \delta(y, z, x) = 0, \tag{1}$$

$$(x, y, z) - \gamma(x, z, y) + (1 - \delta)(y, z, x) = 0, \tag{2}$$

$$(x, x, x) = 0, \tag{3}$$

where the associator $(x, y, z) = (xy)z - x(yz)$ and γ, δ are integers satisfying $\gamma^2 - \delta^2 + \delta = 1$. If we were interested only in algebras, then γ, δ could be taken to be any elements in the base field satisfying $\gamma^2 - \delta^2 + \delta = 1$.

THEOREM 1. *A ring of (γ, δ) type is anti-isomorphic to a ring of $(-\gamma, 1-\delta)$ type.*

For proof observe that if we interchange x and y in (1) and reverse the order of the elements, the result is (2) for a ring of type $(-\gamma, 1-\delta)$. Similarly, reversing the order of the elements in (2) and interchanging x and y yields (1) for a ring of type $(-\gamma, 1-\delta)$.

COROLLARY. *A ring of type $(-1, 0)$ is anti-isomorphic to a ring of type $(1, 1)$.*

Thus from now on it is necessary to consider only the rings of type $(1, 1)$. It is assumed throughout the characteristic is not 2.

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In an appendix at the end of the paper we include a recent communication received by the author from Erwin Kleinfeld. The results of the appendix prove for rings the results of [2] which were proved for algebras.

2. **Identities and power-associativity.** For a ring R of type (1, 1) the defining identities (1), (2) may be written as

$$(x, y, z) + (y, z, x) + (z, x, y) = 0, \tag{4}$$

$$(x, y, z) = (x, z, y). \tag{5}$$

Note that (4) implies $3(x, x, x)=0$ and thus (3) is necessary only when R has characteristic 3.

The following identities hold for any ring R.

$$(xy, z) + (yz, x) + (zx, y) = (x, y, z) + (y, z, x) + (z, x, y),$$

$$(xy, z) - x(y, z) - (x, z)y = (x, y, z) - (x, z, y) + (z, x, y),$$

where the commutator $(x, y)=xy-yx$. In an algebra of type (1, 1) these become

$$(xy, z) + (yz, x) + (zx, y) = 0, \tag{6}$$

$$(xy, z) = x(y, z) + (x, z)y + (z, x, y). \tag{7}$$

Another identity which holds in any ring is

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z. \tag{8}$$

Interchange y and z to obtain $(wx, z, y)-(w, xz, y)+(w, x, zy) =w(x, z, y)+(w, x, z)y$. Subtract this from (8) and use (5) to obtain $(w, y, xz)-(w, z, xy)+(w, x, (y, z))=(w, x, y)z-(w, x, z)y$. Since $(w, y, xz)=(w, y, zx)-(w, y, (z, x))$, we may write this as

$$\begin{aligned}(w, y, zx) - (w, z, xy) + (w, x, (y, z)) - (w, y, (z, x))&\\ = (w, x, y)z - (w, x, z)y.&\end{aligned} \tag{9}$$

A nonempty subset S of a ring R will be called *abelian*[2] if multiplication of elements of S is both commutative and associative; that is, if $(S, S)=(S, S, S)=0$. Relation (3) implies that every subset with one element is an abelian subset. Every abelian subset is contained in a maximal abelian subset—by Zorn's lemma.

[2] This definition of abelian subset is due to the referee.

THEOREM 2. *Let R be a ring of type* $(1, 1)$ *with characteristic not* 2. *Then any maximal abelian subset S of R is a subring.*

By (3), (4), and (5), in order to show that an element r of R is in S, it suffices to show that $(r, S)=(r, r, S)=(S, r, r,)=(r, S, S)=(S, S, r)=0$. To prove the theorem we need to show $SS\subseteq S$. Thus we need to show $(SS, S)=(SS, SS, S)=(S, SS, SS)=(SS, S, S)=(S, S, SS)=0$. The fact that S is a group with respect to addition follows from the definition of S.

Let v, w, x, y, z be any elements of S. From (7) it follows that $(SS, S)=0$. By (9), $(w, y, zx)=(w, z, xy)$. Permuting x, y, z cyclically, $(w, z, xy)=(w, x, yz)$ and then (8) gives $(wx, y,z)=0$, or $(SS, S, S)=0$. Since $(w, z, xy)=(w, z, yx)-(w, z, (y, x))$, $(w, z, xy)=(w, z, yx)$. Then by (4), $(x, y, zw)+(y, zw, x)=-(zw, x, y)=0$, $(x, y, zw)=-(y, x, zw)=-(y, z, wx)=-(y, z, xw)$. Repeating this process we get $(x, y, zw)=-(y, z, xw)=(z, x, yw)=-(x, y, zw)$. Hence, $(S, S, SS)=0$.

Replace w in (9) by vw and use the above results to obtain $(vw, y, zx)=(vw, z, xy)=(vw, x, yz)$. Then (8) with vw instead of w yields $((vw)x, y, z)=0$. Also, (7) implies $(vw, xy)=0$ and $((vw)x, y)=0$. Furthermore (7) implies $(x(vw), y)=0$ and $(vw, x(yz))=0$ where we have used the fact that $((vw)x, y, z)=0$ implies $(x(vw), y, z)=0$. The above results used in (7) now give $(x(yz), vw)=0=(vw, x, yz)$. Then formula (5) implies $(SS, SS, S)=0$. The fact that $(S, SS, SS)=0$ now follows from (4). This completes the proof of Theorem 2.

COROLLARY 1. *Every ring R of type* $(1, 1)$ *is power-associative.*

It was observed earlier that every subset with only one element is abelian and by Theorem 2 is contained in a maximal abelian subring S. Since all powers of this element are in S, the result of Corollary 1 follows.

COROLLARY 2. *Every commutative subring of a ring of type* $(1, 1)$ *is associative.*

The proof follows immediately from (7).

COROLLARY 3. *Two elements x, y of R generate a commutative, associative subring S if and only if* $(x, y)=(x, y, y)=(y, x, x)=0$.

From (4) with $z=y$, $(x, y, y)+2(y, y, x)=0$, so $(x, y, y)=0$ implies $(y, y, x)=0$. Similarly, $(x, x, y)=0$. Thus the two elements x, y form an abelian subset and by Theorem 2 the subring they generate must be commutative and associative. The other half of the corollary is obvious.

3. **Decomposition relative to an idempotent.** The decomposition relative to an idempotent follows from

LEMMA 1. *Let R be a ring of type* (1, 1) *of characteristic not* 2 *and with an idempotent e. Then* $(x, e, e) = (e, x, e) = (e, e, x) = 0$ *for each x of R.*

By relations (4) and (5) we have $(x, e, e) = -2(e, e, x)$. From (8) we obtain four relations by replacing w, z, y, x in turn by x and replacing the other three quantities by e. The first is $(xe, e, e) = (x, e, e)e$. Our previous remark gives $(e, e, xe) = (e, e, x)e$. The second relation obtained from (8) is $(e, e, ex) = e(e, e, x)$ and thus $(ex, e, e) = e(x, e, e)$. These facts shall be used in the last two relations we get from (8). Thus $(e, x, e) - (e, ex, e) + (e, e, xe) = e(e, x, e) + (e, e, x)e$ becomes $(e, e, x) - e(e, e, x) + (e, e, x)e = e(e, e, x) + (e, e, x)e$. That is, $(e, e, x) = 2e(e, e, x)$. Also $(ex, e, e) - (e, xe, e) + (e, x, e) = e(x, e, e) + (e, x, e)e$ gives $(e, e, x) = 2(e, e, x)e$. If we set $a = (e, e, x)$, we have $a = 2ea = 2ae$ and consequently $(a, e) = 0$. Using this fact in (7), $(ee, a) = 0 = (a, e, e) = (ae)e - ae$. Since $a = 2ae$, $ae = 2(ae)e$ and since $(ae)e = ae$, we have $ae = 0$, $a = (e, e, x) = 0$. The remark at the beginning of the proof gives $(x, e, e) = 0$. Finally, $(e, x, e) = 0$ then follows from (5).

THEOREM 3. *Any element x in R, a ring of type* (1, 1) *whose characteristic is not* 2, *may be written uniquely as a sum* $x = x_{11} + x_{10} + x_{01} + x_{00}$ *where* x_{ij} *is in* $R_{ij}(e)$, *the set of all elements y in R such that* $ey = iy$ *and* $ye = jy$.

The results of Lemma 1 could be given in terms of right and left multiplications as $R_e^2 = R_e$, $L_e^2 = L_e$, and $R_eL_e = L_eR_e$. These facts imply the decomposition of Theorem 3 just as they do for associative rings.

The multiplicative properties of the sets R_{ij} are, however, different from those in the case of associative rings.

LEMMA 2. *Let R be a ring of type* (1, 1) *whose characteristic is not* 2 *and with idempotent e. If x is in* $R_{ij}(e)$ *and y is in* $R_{km}(e)$ *and if* $s = j - k$, $t = m - i$, *then*

$$(xy)e = (s + m)xy, \qquad (yx)e = (t + j)yx, \tag{10}$$

$$e(xy) = (s + i)xy + tyx, \qquad e(yx) = (t + k)yx + sxy, \tag{11}$$

$$e(xy)e = (s + m)(s + i)xy + (s + m)tyx, \tag{12}$$

$$(i - k)(j - k)xy = 0 = (s + m)(s + m - 1)xy. \tag{13}$$

Since (5) implies $(x, e, y) = (x, y, e)$, we have $jxy - kxy = (xy)e - mxy$. This gives the first relation of (10). Similarly $(y, e, x) = (y, x, e)$ gives the other half of (10). By (4), $(x, y, e) + (y, e, x) + (e, x, y) = 0$ or

$(xy)e-mxy+myx-iyx+ixy-e(xy)=0$. Using (10) we obtain the first part of (11) and the second part follows by interchanging the roles of x and y. Multiply (10) by e on the left and then use (11) to obtain (12). Right multiplication of (11) by e implies $e(xy)e = (s+i)(s+m)xy + t(t+j)yx$. Compare this with (12) to get $t(t+j-s-m)yx=0$ which may be written $(m-i)(k-i)yx=0$. Interchange x and y to get the first part of (13). Finally, multiply the first relation of (10) by e on the right and use the fact that $(xy, e, e)=0$ to get the second part of (13).

The relations of Lemma 2 imply the following theorem.

THEOREM 4. *Let R be a ring of type* (1, 1) *whose characteristic is not* 2 *and with idempotent e. Then $R_{ij}R_{km}=0$ if $j\neq k$ and $R_{ij}R_{jm}\subseteq R_{im}$ except for the following*:

$$R_{11}R_{10} \subseteq R_{10} + R_{00}, \qquad R_{00}R_{01} \subseteq R_{01} + R_{11},$$
$$R_{10}R_{11} \subseteq R_{00}, \qquad R_{01}R_{00} \subseteq R_{11}.$$

Also $xy-yx$ is in R_{10} for every x in R_{11} and y in R_{10} and $xy-yx$ is in R_{01} for every x in R_{00} and y in R_{01}.

These relations follow in a straightforward manner from Lemma 2. We illustrate the proof by considering x in R_{11}, y in R_{10}. Then $(xy)e=0=(yx)e$, $e(xy)=xy-yx$, $e(yx)=0$ follow from (10) and (11). These imply $R_{11}R_{10}\subseteq R_{10}+R_{00}$ and $R_{10}R_{11}\subseteq R_{00}$. Moreover, $(xy-yx)e=0$ and $e(xy-yx)=xy-yx$ so $xy-yx$ is in R_{10}. The remainder of the proof is obtained in a similar manner.

The fact that the results of Theorem 4 are the best obtainable is illustrated by an example. Let R be an algebra over any field F with a basis e, x, y, w. Assume that e is an idempotent, x is in R_{11}, y in R_{10}, and w in R_{00}. Take all products to be zero except $xy=yx=\alpha w$ where α is any nonzero element of F. It is not hard to see that R is of type (1, 1) and clearly $R_{11}R_{10}=R_{10}R_{11}=R_{00}$.

4. **Construction of an ideal.** Let $G_0=R_{10}R_{11}$ and $G_1=R_{01}R_{00}$. By Theorem 4, G_i is a subset of R_{ii}.

LEMMA 3. *For any element a in G_i, $a^2=0$. Also G_i is an ideal of R_{ii} for $i=0, 1$.*

Let a be in G_0 so that $a=xy$ with x in R_{10}, y in R_{11}. Using the fact that $ay=ya=0$, (5) with $z=a$ yields $a^2=(xa)y$. Since $x^2=ax=0$, (4) implies $(y, x, x)+2(x, x, y)=0$ or $2xa=(yx)x$. But Theorem 4 gives $(yx)x=0$ and so $xa=0$, $a^2=0$. Now let z be any element of R_{00}. By (4) and Theorem 4, $(xy)z=z(xy)$. Relation (5) implies $(xy)z=(xz)y$

which is in G_0 by Theorem 4. This completes the proof of the part of the lemma concerning G_0. The other half is obtained by interchanging subscripts 0 and 1 in the above argument.

THEOREM 5. *The set* $B=G_1+G_0+H$ *with* $H=R_{01}G_1+R_{10}G_0+G_1R_{10}+G_0R_{01}$ *is an ideal of* R.

By Lemma 3 and Theorem 4 we have $(G_1+G_0)R$ and $R(G_1+G_0)$ are contained in B. Consider HR_{11}. Since $(R_{01}, G_1, R_{11})=(R_{01}, R_{11}, G_1)$, $(R_{01}G_1)R_{11}$ is in $R_{01}G_1$. Also, $(G_0, R_{01}, R_{11})=(G_0, R_{11}, R_{01})$ which implies $(G_0R_{01})R_{11}$ is in G_0R_{01}. These facts together with the multiplicative relations of Theorem 4 yield $HR_{11}\subseteq B$. Similarly, $(R_{11}, R_{10}, G_0)=(R_{11}, G_0, R_{10})$ gives $R_{11}(R_{10}G_0)$ in B, and $(R_{11}, G_1, R_{10})=(R_{11}, R_{10}, G_1)$ gives $R_{11}(G_1R_{10})$ in B. It follows that $R_{11}H\subseteq B$. By interchanging the subscripts 0 and 1, we obtain the proof of the facts $HR_{00}\subseteq B$, and $R_{00}H\subseteq B$.

Next look at the products HR_{10}. Using (5), $(R_{01}, R_{00}, R_{10})=(R_{01}, R_{10}, R_{00})$ implies that $(R_{01}R_{00})R_{10}$ is in R_{00}; that is $G_1R_{10}\subseteq R_{00}$. Then $(R_{01}, G_1, R_{10})=(R_{01}, R_{10}, G_1)$ gives $(R_{01}G_1)R_{10}\subseteq G_1$. Theorem 4 implies that $(R_{10}G_0)R_{10}=(G_1R_{10})R_{10}=0$. The products of the form $(G_0R_{01})R_{10}$ are also in B. This is obtained from $(G_0, R_{01}, R_{10})=(G_0, R_{10}, R_{01})$. The set $R_{10}H$ is contained in B. This part of the proof, as the preceding part, is made by using the distributive law and considering the various components. Since $(R_{10}, R_{01}, G_1)=(R_{10}, G_1, R_{01})$, $R_{10}(R_{01}G_1)$ is in $G_1+G_0R_{01}$. Theorem 4 gives $R_{10}(R_{10}G_0)=0$, $R_{10}(G_1R_{10})\subseteq R_{10}G_0$. Earlier in this proof we had $G_1R_{10}\subseteq R_{00}$. Interchange the subscripts 0 and 1 to get $G_0R_{01}\subseteq R_{11}$. Then $R_{10}(G_0R_{01})\subseteq G_0$.

The proof that $HR_{01}\subseteq B$ and $R_{01}H\subseteq B$ is made by interchanging the subscripts 0, 1 in the preceding paragraph. This completes the proof of Theorem 5.

5. **Simple rings.** Let R be a simple ring of type (1, 1) and suppose R has an idempotent e. By Lemma 3, e is not in the ideal B constructed in the preceding section. A simple ring by definition has no proper ideals so it follows that $B=0$, $G_0=G_1=0$. Then the components of the decomposition of a simple ring with idempotent have the same multiplicative properties as an associative ring with idempotent. We can now use the material of §3 of [2] since this material depends only on having relations (4) and (5) and an associative type decomposition. The result obtained is stated in the following theorem.

THEOREM 6. *Let* R *be a simple ring of type* (1, 1) *with idempotent* e *and characteristic not* 2. *Then* R *is an associative ring or* e *is the unity element of* R.

If R is a simple finite dimensional algebra over a field F then, since R is power-associative, R must contain an idempotent e and, as is proved in [2, §3], R is associative, or R must contain an absolutely primitive unity element e.

Appendix. In a letter dated February 15, 1958, Erwin Kleinfeld sent the following results to the author. These results prove the results of [2] for rings and we now reproduce Kleinfeld's work.

We consider rings R of (γ, δ) type, $\delta \neq 0, 1$. By this we mean rings satisfying

$$f(x, y, z) = \gamma(x, z, y) + \delta(y, z, x) + (z, x, y) = 0, \tag{1}$$

$$g(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0, \tag{2}$$

for all x, y, z in R and such that $\alpha x=0$ implies $x=0$, whenever $\alpha=\delta$, $\delta-1$, or 3.

We also use the fact that the identity

$$\begin{aligned} h(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) \\ -(w, x, y)z = 0 \end{aligned} \tag{3}$$

holds in any ring.

Lemma 1. *If $(n, R, R)=0$, then n is in the nucleus N of R.*

Proof. From $0=f(n, y, z)-g(n, y, z)$ follows $(\delta-1)(y, z, n)=0$, so that $(y, z, n)=0$. But then $0=f(n, y, z)$ implies $(z, n, y)=0$ and so n is in N.

Lemma 2. (i) $(N, (R, R, R))=0$, (ii) $(N, R)\subseteq N$, (iii) $T=\{t\in R \mid (N, R)t=0\}$ *is an ideal of R.*

Proof. From $0=-g(zn, x, y)+h(x, y, z, n)-h(y, z, n, x)+g(nx, y, z)+h(z, n, x, y)-h(n, x, y, z)$ follows $(n, (x, y, z))=0$, thus proving (i).

From $0=-f(z, xn, y)+\gamma h(z, y, x, n)+h(y, z, x, n)+f(z, x, y)n+\delta(n, (x, y, z))+\delta h(n, x, y, z)$ follows $\delta((n, x), y, z)=0$, so that $((n, x), y, z)=0$. From this and Lemma 1 follows (ii). Because of (ii), T is obviously a right ideal. But because of (ii) $((N, R), R)\subseteq(N, R)$so that $((N, R), R)T=0$. Since $R(N, R)T=0$, we must have $(N, R)RT=0$, proving (iii).

Theorem. *If R is simple then either $(N, R)=0$ or $N=R$.*

Proof. Assume $(N, R)\neq 0$. Since T is an ideal of R, either $T=R$ or $T=0$. If $T=R$, $0=(N, R)R$; hence $S=\{s\in R \mid sR=0\}$ is an ideal of R and is not zero. But then $S=R$ and $RR=0$, making R associative or $N=R$. Suppose then that $T=0$. If n is in N, we easily verify that

$(n, ab) = a(n, b) + (n, a)b$. Then $(n, w(x, y, z)) = (n, w)(x, y, z)$ using (i). But $(n, w)(x, y, z) = (x, y, z)(n, w)$ because of (i) and (ii), while $(x, y, z)(n, w) = (n, (x, y, z)w)$. Therefore

$$(n, w(x, y, z)) = (n, (x, y, z)w). \tag{4}$$

Then $0 = (n, h(x, y, z, x)) = (n, x(y, z, x) + (x, y, z)x)$ using (i). On the other hand $(n, (x, y, z)x) = (n, x(x, y, z))$ because of (4). Thus $(n, x(y, z, x) + x(x, y, z)) = 0$. Comparison of this last identity with $(n, xg(x, y, z)) = 0$ yields

$$(n, x(z, x, y)) = 0. \tag{5}$$

Then $0 = (n, xf(y, z, x))$ and (5) lead to

$$(n, x(x, y, z)) = 0, \tag{6}$$

and since $(n, x(y, z, x) + x(x, y, z)) = 0$,

$$(n, x(y, z, x)) = 0. \tag{7}$$

But $(n, h(x, z, x, y)) = 0$, and (5) imply $(n, (x, z, x)y) = 0$, and then $(n, y(x, z, x)) = 0$ and $(n, y)(x, z, x) = 0$. Thus (x, R, x) is in T so $(x, R, x) = 0$. Similarly one can use (6) and (7) to obtain $(x, x, R) = 0$ and $(R, x, x) = 0$. These and (2) imply $3(x, y, z) = 0$ through linearization, so that $(R, R, R) = 0$ and $N = R$. This completes the proof.

Now if R is simple and not associative we have $(N, R) = 0$. Thus $N = C$, the center of R. If e is any idempotent in R then from the Pierce decomposition one obtains that e is in N. But then e is in C and eR is an ideal of R containing $e \neq 0$. Thus $eR = R$ and e is the identity element of R.

References

1. A. A. Albert, *Almost alternative algebras*, Portugal. Math. vol. 8 (1949) pp. 23–36.

2. L. A. Kokoris, *On a class of almost alternative algebras*, Canad. J. Math. vol. 8 (1956) pp. 250–255.

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