$+0\cdot t+0\cdot 0=2t+0+2t+0=0$. By direct computation it can be checked that R is (3, 3)-distributive.

This example shows that the hypothesis of associativity in Theorem 7 is necessary, since R cannot be isomorphic to $[R-(0)]\oplus(0)$. Since $(0)=\{0, 2t\}$, the additive group of $[R-(0)]\oplus(0)$ is the four group while R^+ is the cyclic group of order 4.

EXAMPLE 3. Again let $R^+=\{0, t, 2t, 3t\}$ be a cyclic group of order 4, and define multiplication by $xy=2t$. Then R is associative and commutative and is an (m, n)-distributive ring if m and n are both odd. Here $(0)=\{0, 2t\}$, so that as in Example 1, the additive group of $[R-(0)]\oplus(0)$ is the four group. Hence R cannot be isomorphic to $[R-(0)]\oplus(0)$, and this example shows that the hypothesis $0\cdot 0=0$ is necessary in Theorem 7.

UNIVERSITY OF WASHINGTON

ON TOTALLY BOUNDED SUBSETS OF SEQUENCE SPACES

CHARLES W. McARTHUR

1. **Introduction.** Let X denote a Banach space with a denumerable biorthogonal basis $\{x(i)\}$, $\{f_i\}$. Cohen and Dunford [2, Theorem 2] show that *a set* $S\subset X$ *is conditionally compact* (or, equivalently, totally bounded) *in* X *if and only if* S *is bounded and* $\lim_n \sum_{i=1}^n f_i(x)x(i)=x$ *uniformly for* $x\in S$. The purpose of this paper is to show that a modified form of the above condition, one which retains a uniform convergence and boundedness requirement, characterizes the totally bounded sets of a class of Banach spaces which includes the class of those Banach spaces having a basis as a proper subclass.

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2. **Definition of A-spaces and preliminaries.** Throughout the paper X will stand for a real Banach space (B-space). Its zero will be written as θ. The set of positive integers we denote by N. A sequence in X will be represented usually by a single letter s and its value at each $i\in N$ by $s(i)$. A sequence s in X will be called *finitely nonzero* if and only if $s(i)\neq\theta$ holds for at most a finite number of $i\in N$. Occasionally when the norm symbol appears in the same expression in different

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senses a subscript will be used for clarity, e.g., $\|x\|_X$ will mean the norm of $x \in X$.

Definition 1. For each $n \in N$ we define a function T_n on the set of sequences in X to the set of sequences in X by $(T_n(s))(i) = \theta$ if $i > n$, and $(T_n(s))(i) = s(i)$ if $i \leqq n$, for each sequence s in X and $i \in N$. The function T_n will be called the nth *truncation operator*.

Definition 2. For each $n \in N$ we define the function K_n on the set of sequences in X to the set of sequences in X by $(K_n(s))(i) = \theta$ if $i \neq n$, and $(K_n(s))(i) = s(i)$ if $i = n$, for each sequence s in X and $i \in N$. The function K_n will be called the nth *selection operator*.

Definition 3. An A-space of X is a real Banach space $W(X)$ whose elements are sequences in X and with addition and scalar multiplication of its elements defined in the usual way for spaces of sequences. Also $W(X)$ satisfies these conditions:

(i) The finitely nonzero sequences in X are elements of $W(X)$.

(ii) There exists $M > 0$ such that for each $n \in N$ the nth truncation operator satisfies $\|T_n(s)\| \leqq M\|s\|$ for all $s \in W(X)$.

(iii) $\operatorname{Lim}_n T_n(s) = s$ for each $s \in W(X)$.

(iv) For each $n \in N$ the nth selection operator satisfies $\|K_n(s)\|_{W(X)} = \|s(n)\|_X$ for all $s \in W(X)$.

Condition (ii) implies that each T_n is continuous. Furthermore, since $K_1 = T_1$ and if $n > 1$, $K_n = T_n - T_{n-1}$ it follows from the continuity of T_n that each K_n is continuous.

Definition 4. For each $n \in N$ we define the nth *projection operator* P_n on $W(X)$ to X by $P_n(s) = s(n)$ for each $s \in W(X)$.

From (iv) of Definition 3 and the continuity of K_n it follows that each projection operator is continuous.

3. **Examples of A-spaces.** Consider the following three well known B-spaces corresponding to an arbitrary B-space X. Let $c_0(X)$ denote the space of null sequences of X with norm given by $\sup \{\|s(i)\| : i \in N\}$ for each null sequence s. Let $a(X)$ represent the space of sequences s in X such that $\sum_{i=1}^{\infty} \|s(i)\| < \infty$ and with the norm of each s given by $\sum_{i=1}^{\infty} \|s(i)\|$. Let $u(X)$ denote the space of sequences s in X such that the series $\sum_{i=1}^{\infty} s(i)$ converges unconditionally and with the norm of each s given by $\sup \{\| \sum_{i \in F} s(i)\| : F \text{ is finite and } F \subset N\}$. Each of these spaces is an A-space as may readily be verified.

If $W(X)$ is an A-space of X with X the B-space of real numbers we shall call $W(X)$ an *A-space of the reals*. We point out that *any B-space with a denumerable basis is isomorphic to an A-space of the reals*. Let X be a B-space with a denumerable basis $\{x(i)\}$. Then

$$b = \{x(i)/\|x(i)\|\}$$

is also a basis for X. Banach notes, [**1**, p. 111], that isomorphic to X is the B-space $W(X, b)$ of sequences of real numbers defined as follows:

$$W(X, b) = \left[s: \text{ the series } \sum_{i=1}^{\infty} s(i)x(i)/\|x(i)\| \text{ converges} \right] \tag{1}$$

with the norm of each $s \in W(X, b)$ defined as

$$\|s\| = \sup \left\{ \left\| \sum_{i=1}^{n} s(i)x(i)/\|x(i)\| \right\| : n \in N \right\}. \tag{2}$$

We omit the simple verification of the fact that $W(X, b)$ is an A-space of the reals.

4. **The Main Theorem.** Lemma 1 below is well known. Lemma 2 was suggested by the referee. Its proof is omitted. We will subsequently use the notation $N(x, r)$ for sphere with center x and radius r.

LEMMA 1. *Suppose X and Y are B-spaces and S is a totally bounded subset of X. If $\{L_n\}$ is a sequence of linear operators on X to Y such that for some $M>0$, $\|L_n\| \leqq M$ for all $n \in N$, and $\{L_n(x)\}$ converges for each $s \in S$, then $\{L_n(x)\}$ converges uniformly on S.*

LEMMA 2. *If $\{B_n\}$ is a sequence of totally bounded subsets of X and S is a subset of X such that* $\lim_{n\to\infty} \sup_{x\in S} \inf_{y\in B_n} \|x-y\| = 0$ *then S is totally bounded.*

LEMMA 3. *Suppose S_i, $i=1, 2, \cdots, n$, are totally bounded subsets of X and that $W(X)$ is an A-space of X. Let*

$$C = \{ s \in W(X) : s(i) \in S_i \text{ if } i \leqq n \text{ and } s(i) = \theta \text{ if } i > n \}.$$

Then C is a totally bounded subset of $W(X)$.

PROOF. Let $\epsilon > 0$ be given. Corresponding to each i, $1 \leqq i \leqq n$, there is a finite number $m(i)$ of elements of X which we label $b_1(i), \cdots, b_j(i), \cdots, b_{m(i)}(i)$ such that

$$S_i \subset \bigcup_{j=1}^{m(i)} N(b_j(i), \epsilon/n). \tag{3}$$

We define a finite subset B of $W(X)$ as the set of all $b \in W(X)$ such that if $i \leqq n$ then $b(i) = b_j(i)$ for some j such that $1 \leqq j \leqq m(i)$ and if $i > n$ then $b(i) = \theta$. We now show that if $s \in C$ there exists $b \in B$ such that $\|s-b\| < \epsilon$. Given $s \in C$ we define $b \in B$ as follows:

If $i \leqq n$, using (3) we choose a $b_j(i)$ such that $\|s(i) - b_j(i)\|_X < \epsilon/n$

and we define $b(i)=b_j(i)$. If $i>n$ we define $b(i)=\theta$. Then since $s-b=\sum_{i=1}^{n} K_i(s-b)$ one has that

$$\|s-b\| \leqq \sum_{i=1}^{n} \|K_i(s-b)\| \leqq \sum_{i=1}^{n} \|s(i)-b(i)\|_X < \epsilon.$$

THEOREM 1. *If $W(X)$ is an A-space of X then a subset S of $W(X)$ is totally bounded if and only if $\lim_n T_n(s)=s$ uniformly on S, where $\{T_n\}$ is the sequence of truncation operators on $W(X)$, and $\{s(i): s\in S\}$ is totally bounded in X for each positive integer i.*

PROOF. If S is a totally bounded subset of $W(X)$ then, by Lemma 1, $\lim_n T_n(s)=s$ uniformly on S. For each $i\in N$ the set $\{s(i): s\in S\}$ is totally bounded being the image of S under the ith projection operator.

To prove the converse we assume that $\lim_n T_n(s)=s$ uniformly on S and $S_i\equiv\{s(i): s\in S\}$ is totally bounded in X for each $i\in N$. For each $n\in N$ let $C_n=[s\in W(X): s(i)\in S_i$ if $i\leqq n$ and $s(i)=\theta$ if $i>n]$. Each $T_n(S)$ is totally bounded since it is a subset of C_n which is totally bounded by Lemma 3. Then using Lemma 2 with $B_n=T_n(S)$ we conclude that S is totally bounded.

COROLLARY 1. *If X has finite dimension and $W(X)$ is an A-space of X then $S\subset W(X)$ is totally bounded if and only if S is bounded and $\lim_n T_n(s)=s$ uniformly on S.*

We now verify that the following corollary due to Cohen and Dunford [2, Theorem 2] follows from Corollary 1.

COROLLARY 2. *If X has a biorthogonal basis $\{x(i)\}$, $\{f_i\}$, then a subset A of X is totally bounded in X if and only if A is bounded in X and $\lim_n \sum_{i=1}^{n} f_i(a)x(i)=a$ uniformly for $a\in A$.*

PROOF. It follows from (1) of §3 that the A-space of the reals corresponding to X and $\{x(i)\}$ is $W(X, b)=\{\{f_i(x)\cdot\|x(i)\|\}: x\in X\}$. The function U defined on $W(X, b)$ onto X by

$$U(s)=\sum_{i=1}^{\infty} s(i)x(i)/\|x(i)\| \text{ for each } s\in W(X, b)$$

is a linear homeomorphism as is its inverse U^{-1} [1, p. 111]. Now $A\subset X$ is totally bounded in X if and only if $U^{-1}(A)$ is totally bounded in $W(X, b)$. By Corollary 1, $U^{-1}(A)$ is totally bounded if and only if it is bounded in $W(X, b)$ and $\lim_n T_n(\{f_i(a)\cdot\|x(i)\|\})=\{f_i(a)\cdot\|x(i)\|\}$ uniformly for $a\in A$. Since U is a linear homeomorphism it follows that A is totally bounded in X if and only if $A=U(U^{-1}(A))$ is bounded in X and $\lim_n U(T_n(\{f_i(a)\cdot\|x(i)\|\}))=\lim_n \sum_{i=1}^n f_i(a)x(i)$ $=a$ uniformly for $a\in A$.

Lemma 4. *If $W(X)$ is an A-space of X such that whenever $S\subset W(X)$ is bounded and* $\lim_n T_n(s)=s$ *uniformly on S it follows that S is totally bounded, then X has finite dimension.*

Proof. Let A be a bounded set in X. By (iv) of Definition 3 $S\equiv\{s\in W(X): s(1)=a$ for some $a\in A$ and $s(i)=\theta$ if $i>1\}$ is bounded in $W(X)$. Since $T_n(s)=s$ for each $n\in N$ and $s\in S$, $\lim_n T_n(s)=s$ uniformly on S. Hence, S is totally bounded in $W(X)$. Now A is the image of S under the projection operator P_1 so A must be totally bounded in X. In particular the unit sphere of X is totally bounded which implies that X has finite dimension.

Corollaries 3, 4, and 5 are immediate consequences of Theorem 1 and Corollary 1. By Lemma 4, condition (ii) of Corollaries 3, 4, and 5 is equivalent to "S is bounded" if and only if X has finite dimension. Corollaries 3 and 5 have been stated by Cohen and Dunford [2] for the case X equal to the reals.

Corollary 3. *A set $S\subset a(X)$ is totally bounded if and only if*

(i) $\epsilon>0$ *implies there is $n(\epsilon)$ such that $\sum_{i=n}^{\infty}\|s(i)\|<\epsilon$ for all $n\geqq n(\epsilon)$ and $s\in S$, and*

(ii) *the set $\{s(i): s\in S\}$ is totally bounded in X for each $i\in N$.*

Corollary 4. *A set $S\subset u(X)$ is totally bounded if and only if*

(i) $\epsilon>0$ *implies there is $n(\epsilon)$ such that* $\sup\{\|\sum_{i\in F} s(i)\|$ *: F is finite and $F\cap[1, n(\epsilon)]=\phi\}<\epsilon$ for all $s\in S$, and*

(ii) *the set $\{s(i): s\in S\}$ is totally bounded in X for each $i\in N$.*

Corollary 5. *A set $S\subset c_0(X)$ is totally bounded if and only if*

(i) $\epsilon>0$ *implies there is $n(\epsilon)$ such that* $\sup\{\|s(i)\|: i\geqq n(\epsilon)\}<\epsilon$ *for all $s\in S$, and*

(ii) *the set $\{s(i): s\in S\}$ is totally bounded in X for each $i\in N$.*

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The Florida State University