B PARACOMPACT DOES NOT IMPLY B^I PARACOMPACT[1]

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As is well known the category of paracompact spaces is important in algebraic topology and the theory of fiber spaces. The following question arises naturally. If a space (Hausdorff space) B is paracompact, is the space of paths B^I ($I=[0, 1]$), with, of course, the compact-open topology, also paracompact? The following simple example answers the question in the negative.

Let X denote the set of real numbers with the half-open interval topology [1]. This now well-known space has the following properties: regular, Lindelöf (hence paracompact [2], hence normal [3]) and totally disconnected. It is also known that $X \times X$ is not normal [1] (hence not paracompact). Since X^I and X are homeomorphic, X^I is paracompact so a slight adjustment must be made to provide the counter-example. Let $C(X)$ denote the cone over X, i.e., in $X \times I$ identify $X \times \{1\}$ to a point, thus obtaining $C(X)$. Then, if $p: X \times I \rightarrow C(X)$ is the identification map, $C(X)$ is topologized by employing the weakest topology which renders p continuous. Since $X \times I$ is Lindelöf and regular, it follows that $C(X)$ is Lindelöf and regular, hence paracompact. What we will show now is that $C(X)^I$ is not paracompact. The idea is the following: X appears in $C(X)$ as a *closed* subset, namely the "base" of the cone. Therefore $X \times X$ appears in $C(X) \times C(X)$ as a *closed* subset and hence $C(X) \times C(X)$ is not paracompact. Thus, if we can imbed $C(X) \times C(X)$ in $C(X)^I$ as a *closed* subset, it will follow that $C(X)^I$ is not paracompact.

We leave to the reader the simple proofs of the following lemmas.

LEMMA. *Let Y denote a space and F: $Y \times I \rightarrow Y$ a contraction of Y to $y_0 \in Y$, i.e., $F_0 = 1$ and $F_1 = y_0$. Then the mapping $\overline{F}$: $Y \rightarrow Y^I$ given by $\overline{F}(y)(s) = F(y, s)$, $0 \leqq s \leqq 1$, $y \in Y$, is an imbedding of Y in Y^I whose image $\overline{F}(Y)$ is closed in Y^I.*

LEMMA. *Let B and Y denote spaces and fix $b \in B$. Furthermore, let $\tilde{B} = \{\omega \in B^I : \omega(1) = b\}$. Let f: $Y \rightarrow \tilde{B}$ be a map such that $f(Y)$ is closed in $\tilde{B}$ (hence in B^I). Define a map f^2: $Y \times Y \rightarrow B^I$ by*

$$f^2(y, y') = f(y) \circ f(y')^{-1}$$

where $\circ$ denotes multiplication of paths. Then, $f^2(Y \times Y)$ is closed in B^I.

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Now, let F: $C(X) \times I \rightarrow C(X)$ denote the usual contraction of $C(X)$ to its vertex, i.e.,

$$F[p(x, t), s] = p((x, t + s - st)) \cdot$$

Applying the above lemmas with $Y = C(X)$, $B = C(X)$, F the contraction of $C(X)$ to its vertex and $f = \bar{F}$, we see that $\bar{F}^2$: $C(X) \times C(X) \rightarrow C(X)^I$ has a closed image, i.e., $\bar{F}^2(C(X) \times C(X))$ is closed in $C(X)^I$. Thus, to complete the proof that $C(X)^I$ is not paracompact, it suffices to show that $\bar{F}^2$ is an imbedding. This fact, however, is immediate as follows: Define a map ϕ: $C(X)^I \rightarrow C(X) \times C(X)$ by setting $\phi(\alpha) = (\alpha(0), \alpha(1))$. Thus $\phi | \bar{F}^2(C(X) \times C(X))$ is the required inverse for $\bar{F}^2$.

THEOREM. *$C(X)$ is paracompact but $C(X)^I$ is not paracompact.*

REMARK. Thus, for example, we see that if one is considering maps f: $X \rightarrow Y$ in the category of paracompact spaces, the usual technique of replacing f by a fiber map may take one outside of this category.

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