A SUFFICIENT CONDITION THAT A MONOTONE IMAGE OF THE THREE-SPHERE BE A TOPOLOGICAL THREE-SPHERE

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1. A continuous transformation of one space onto another is called monotone provided the complete inverse set for each point of the image space is connected. A monotone image of a circle is a simple closed curve or a point. A monotone image of a 2-sphere is a configuration known as a cactoid, i.e. a peano space in which every true cyclic element is a topological 2-sphere. R. L. Moore has shown that if a monotone transformation of a 2-sphere has the additional property that no inverse set separates the 2-sphere, then the image space is again a topological 2-sphere or a point [3]. In the case of the three-sphere, $S^3$, as one would expect, the situation is more complicated and extra conditions need to be imposed if the image space is to be expected to look like an $S^3$.

A recent example of R. H. Bing [1] shows that if a monotone transformation on $S^3$ has the property that for each point of the image the complement of the inverse image is an open 3-cell, the image may not be a topological $S^3$, thus answering a long standing conjecture. By studying this example and profiting by conversations with Professor Bing the author was led to the following theorem.

2. Theorem 1. Let $M = f(S^3)$, where $f$ is a monotone, continuous map such that (i) if $Y = \{ y \in M | f^{-1}(y) \text{ does not reduce to a point} \}$, then given $y \in \overline{Y}$, and $\epsilon > 0$, there is a topological 2-sphere $K$ in $S(y, \epsilon)$ separating $y$ and $M \setminus S(y, \epsilon)$ such that $K$ does not meet $\overline{Y}$. Then $M$ is a topological 3-sphere.2

Proof. Let $\epsilon_1 > \epsilon_2 > \cdots \to 0$ and $\sum \epsilon_i < + \infty$. The set $\overline{Y}$ is totally disconnected. Hence $\overline{Y} = Y_1 \cup \cdots \cup Y_n$ where $Y_i$ is closed,3 $Y_i \cap Y_j = \emptyset$ and $\delta(Y_i) < \epsilon_i/4$. Suppose $\eta_i = \min \rho [Y_i, Y_j]$, $i \neq j$. De-

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2 $\overline{Y}$ = closure of $Y$.

3 $\delta(Y)$ represents the diameter of $Y$, $S(y, \epsilon)$ is the set of points each of whose distance from $y$ is less than $\epsilon$. The symbol $\rho$ represents the metric of the space concerned. It will be clear whether $\rho$ refers to $M$ or $S^3$ by noting in which space the sets are given.

If $K$ is a topological 2-sphere in $M \setminus \overline{Y}$, the complement of $\overline{Y}$ in $M$, $\text{Int } K = f[\text{Int } f^{-1}(K)]$.  

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fine $\epsilon' = \min \left( \epsilon_i/4, \eta_i/3 \right)$. A finite number of topological 2-spheres $K'_1, \cdots, K'_{m_1}$ are found, by use of (i), such that for $i = 1, \cdots, m_1$,

\begin{enumerate}
  \item $\delta(K'_i) < \epsilon'$;
  \item $\cup \text{Int } K'_i \supset \overline{Y}$;
  \item $K'_i \cap \overline{Y} = \emptyset$.
\end{enumerate}

The first set of operations is designed to replace the spheres $K'_1, \cdots, K'_{m_1}$ by a set $\tilde{K}'_1, \cdots, \tilde{K}'_{m_1}$ that enjoy properties similar to (1), (2), (3) and the further requirement

\begin{enumerate}
  \item[$(4)$] $\tilde{K}'_i \cap \tilde{K}'_j = \emptyset$, \quad $i \neq j$.
\end{enumerate}

The set of spheres $\tilde{K}'_1, \cdots, \tilde{K}'_{m_1}$ may be found as follows. Since \( f^{-1} \) is topological on $K'_i$, $L_i = f^{-1}(K'_i)$ is a topological 2-sphere. Since $\rho[L_i, f^{-1}\overline{Y}] > 0$, we may apply the Bing approximation theorem [2] to find a polyhedral 2-sphere $P_i$ as near $L_i$ as we please so that $P_i$ contains in its interior precisely those components of $f^{-1}\overline{Y}$ that are interior to $L_i$. By doing this for each $i$, we obtain a set of polyhedral 2-spheres

\[ P_1, \cdots, P_{m_1}. \]

It may be supposed further that $P_i \cap \overline{P}_j$ is a finite collection (possibly null) of pairwise disjoint simple closed curves, none of which may be removed by an arbitrarily small deformation of $P_i$ or $P_j$. In addition,

\begin{enumerate}
  \item[$(2')$] $\cup \text{Int } P_i \supset f^{-1}(\overline{Y})$.
  \item[$(3')$] $P_i \cap f^{-1}(\overline{Y}) = \emptyset$.
\end{enumerate}

We first describe how to find a set of polyhedral 2-spheres $\tilde{P}_1, \cdots, \tilde{P}_{m_1}$ such that conditions $(2')$, $(3')$ and the following hold

\begin{enumerate}
  \item[$(4')$] $\tilde{P}_i \cap \tilde{P}_j = \emptyset$.
\end{enumerate}

Suppose $C_1, \cdots, C_q$ are the components of $P_1 \cap P_2$. If $q = 1$, let $C_1$ divide $P_1$ into $U_1, V_1$ and $C_1$ divide $P_2$ into $U_2, V_2$. Then $P_1$ and the closure of the component ($V_2$ say) of $P_2 \setminus C_1$ in the exterior of $P_1$ together with the appropriate disk ($U_1$ or $V_1$) gives a pair of 2-spheres $P_1, P'_2$ that covers the same part of $f^{-1}(\overline{Y})$ that $P_1 \cup P_2$ does, neither $P_1$ nor $P'_2$ meets $f^{-1}(\overline{Y})$ and, by a slight deformation $P_1 \cap P'_2 = \emptyset$.

If $q > 1$, at least one of $C_1, \cdots, C_q$, say $C_1$, will not separate $C_2, \cdots, C_q$ on $P_1$. (Of course $C_1$ may separate $C_2, \cdots, C_q$ on $P_2$, but that is irrelevant.) By replacing $P_2$ by 2 new polyhedral 2-spheres meeting along a disk on $P_1$, we again have covered the same part of
$f^{-1}(\overline{V})$ and by a pair of slight deformations obtain 3 polyhedral 2-spheres

$$P_1, P'_2, P''_2$$

such that the number of components of $P_1 \cap P'_2$ or $P_1 \cap P''_2$ is less than $q$.

Continuing, we obtain, after a finite number of such operations a collection of polyhedral 2-spheres

$$\bar{P}_1, \ldots, \bar{P}_{p_1}$$

such that

(2'') $\bigcup \text{Int } \bar{P}_i \supseteq f^{-1}(\overline{V})$.

(3'') $\bar{P}_i \cap f^{-1}(\overline{V}) = \emptyset$.

(4'') $\bar{P}_i \cap \bar{P}_j = \emptyset, \ i \neq j$.

Define $\bar{K}_i = f(\bar{P}_i)$. We note that under the steps made in forming $\bar{P}_i$, or, correspondingly, $\bar{K}_i$, that the diameters of the spheres replacing $K_j$ may be greater than that of $K_j$. However, since $\varepsilon'_1 < \eta_1(1/3)$, the definition of $\eta_1$ and the triangle inequality show that $\delta(\bar{K}_i) < 3\varepsilon_1/4 < \varepsilon_1$. Hence $\bar{K}_1, \ldots, \bar{K}_{p_1}$ satisfy

(1) $\delta(K_i) < \varepsilon_1$;

(2) $\bigcup \text{Int } K'_i \supseteq \overline{V}$;

(3) $\bar{K}_i \cap \overline{V} = \emptyset$;

(4) $\bar{K}_i \cap \bar{K}_j = \emptyset, \ i \neq j$.

To $\varepsilon_2 > 0$, write $Y_i = Y_{i,1} \cup \cdots \cup Y_{i,n_2}$, where $Y_{i,j}$ is closed, $Y_{i,j} \cap Y_{i,j'} = \emptyset$ and $\delta(Y_{i,j}) < \varepsilon_2/4$. Put

$$\eta_2 = \min \rho[Y_{i,j}, Y_{i',j'}], \rho \left[ Y_{i,j}, \bigcup_{i=1}^{n_1} \bar{K}_i \right].$$

Let $\varepsilon'_2 < \varepsilon_2/4, \eta_2/3$. By the hypotheses (i) there is a finite collection of topological 2-spheres in $M, K^2_1, \ldots, K^2_{m_2}$ such that

(1) $\delta(K^2_i) < \varepsilon'_2$:

(2) $\bigcup \text{Int } K^2_i \supseteq \overline{V}$:

(3) $K^2_i \cap \overline{V} = \emptyset$.

By the choice of $\varepsilon'_2$, $K^2_i \cap \bar{K}^2_j = \emptyset$. By modifications of the $K^2_1, \ldots, K^2_{m_2}$ precisely as above at the first stage we arrive at another set of spheres $\bar{K}^2_1, \ldots, \bar{K}^2_{p_2}$ such that
The general step is now clear. To $\varepsilon > 0$ we find a finite set of topological 2-spheres $K_1, \ldots, K_p$ such that

\begin{align*}
(1) & \quad \delta(K_i) < \varepsilon, \\
(2) & \quad \bigcup_{i=1}^{p} \text{Int } K_i \supset \bar{Y}, \\
(3) & \quad K_i \cap \bar{Y} = \emptyset, \\
(4) & \quad K_i \cap K_j = \emptyset, \quad i \neq j \\
(5) & \quad K_i \cap K_j = \emptyset.
\end{align*}

3. Let $F_1', \ldots, F_p'$ be $p_1$ disjoint cubes (topological 2-spheres) with centers on the $x$-axis and faces parallel to the co-ordinate planes. We take the cubes congruent to one another for convenience. Let $F_1^2, \ldots, F_p^2$ be a similar set of cubes of smaller size so that

\[ F_i^2 \subset \text{Int } F_i' \]

if and only if

\[ K_i^2 \subset \text{Int } K_i'. \]

Continuing, for each $n$ we have

\[ F_1^n, \ldots, F_p^n \]

a collection of pairwise disjoint cubes so that

\[ F_i^n \subset \text{Int } F_i^{n-1} \]

if and only if

\[ K_i^n \subset \text{Int } K_i^{n-1}. \]

Without loss we may require that $\delta(F_i^n) < 1/n$. 

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The following lemma is stated without proof.

**Lemma.** If $Q_1, \ldots, Q_n$ are disjoint polyhedral 2-spheres in $S^3$, no one interior to any other, and if $Q_0$ is a large polyhedral cube containing $Q_1, \ldots, Q_n$ in its interior, the closed domain bounded by $Q_0, Q_1, \ldots, Q_n$ is tame. Further, any two domains so formed in this way are homeomorphic.

4. Let $P^0$ be a large cube in $S^3$ containing $P_1, \ldots, P_{p_1}$ in its interior. Then $K^0 = f(P^0)$ is a 2-sphere in $M$ containing $K'_1, \ldots, K'_{p_1}$ in its interior. Let $M_0$ be the region in $M$ exterior to $K^0$. Then $M_0$ is homeomorphic to $S^3 \setminus \text{Int } P^0$ under $f^{-1}$. Hence there is a homeomorphism $h_0$ from $M_0$ to $S^3 \setminus \text{Int } F_0$. Let $M_1 = \text{region in } M \text{ bounded by } K^0 \cup \bigcup_{i=1}^{p_1} K'_i$. Then, by the lemma, $M_1$ is homeomorphic to the region in $S^3$ bounded by $P^0 \cup \bigcup_{i=1}^{p_1} F'_i$. Let $h_1$ be a homeomorphic extension of $h_0$ from $M_0 \cup M_1$ to $M_0 \cup M_1$. The next step is similar, except that $M_2$ is a union of a finite number of regions bounded by the sets

$$
\bigcup_{i=1}^{p_1} K'_i \cup \bigcup_{i=1}^{p_2} K'_i.
$$

However, these regions are in 1-1 correspondence with the number of regions bounded by

$$
\bigcup_{i=1}^{p_1} F'_i \cup \bigcup_{i=1}^{p_2} F'_i,
$$

hence, by the lemma, the extension of $h_1$ from $M_0 \cup M_1 \cup M_2$ can be carried out.

Continuing, a sequence of homeomorphisms $h_0, h_1, h_2, \ldots$ is defined so that each is an extension of the preceding and $h(x) = h_n(x)$ maps $M \setminus \overline{V}$ homeomorphically onto the complement of a Cantor set $X$ in $S^3$.

Since nested sequences of connected sets in $M \setminus \overline{V}$ correspond to nested sequences of connected sets in $S^3 \setminus X$, it is easy to see that $h$ and $h^{-1}$ are both uniformly continuous, hence the extension $\tilde{h}$ of $h$ carries $M$ homeomorphically onto $S^3$.

**References**

1. R. H. Bing, *A decomposition of $E^3$ into points and tame arcs such that the decomposition space is topologically different from $E^3$*, Ann. of Math. vol. 65, no. 3 (1957) pp. 484-500.
