CLASSIFICATION OF FINITE 2-COMPLEXES

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We shall set up, for finite 2-complexes, a complete system of topological invariants, partly numerical and partly of order. The essential part of the classification problem is the characterization of the neighborhoods of the points. The singular points constitute a linear graph whose complement is a collection of bounded 2-manifolds. The knowledge of local structure then enables us to describe the manner in which the boundaries of the manifolds are woven into the singular graph to reassemble the complex.

At first, we shall suppose that we are given a connected finite simplicial 2-complex, K, wherein every edge is a face of some 2-simplex. (Later, the last restriction will be removed.) Let P be an arbitrary point of K, and let st(P) be the star of the open simplex containing P. There are four possibilities for this neighborhood of P. (1) St(P) *is homeomorphic to the euclidean plane.* Then P is called *regular.* The points of K which are not regular are called *singular.* (2) St(P) *is topologically equivalent to the space obtained by identifying the x-axes of a certain* n $(\neq 2)$ *copies of the closed euclidean half-plane* $y \geqq 0$. Then P is called *line-singular* and a neighborhood, or space, of this sort is a *book.* (3) St(P) *is topologically equivalent to the space obtained by identifying the origins of a certain* m (>1) *copies of the euclidean plane.* Then P is called a *conical* point, and a neighborhood of this sort is a *cone.* In both cases, (2) and (3), the regular part of st(P) falls into components called *leaves* of the book or cone, respectively. (4) A singular point which is neither conical nor line-singular is a *node.* In this case there are certain singular edges with P as a vertex. The regular part of st(P) falls into components each of which is a cone leaf or is, topologically, an open triangle with P as vertex and with two singular edges, with P as vertex, which may be distinct or they may coincide. If the edges are distinct, the component is called a *fan,* if they coincide, the component is called a *cornet.* Thus, *the neighborhood* st(P) *of a node* P *consists of a certain number* $(\geqq 0)$ *of leaves of a cone and a certain number of fans and cornets.* To specify st(P) one need only specify the number of cone leaves, the number of singular edges with P as vertex, and the number of fans and/or cornets and specify which singular edges belong to which fans and/or cornets. There are obviously restrictions: there must be some fans or

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cornets in st(P), otherwise P would be conical or regular; there cannot be just fans assembled like a book, since then P would be line-singular; there cannot be just one cornet alone, since then P would be regular; the edge of a fan or cornet having P as vertex cannot be the face of exactly two 2-cells since otherwise that edge would not be singular.

The singular points of K constitute a linear graph, S, the *singular graph* of K. S is, in general, a quite arbitrary finite 1-complex: it is not necessarily connected; it may have points as components (these are the conical points of K); and some components may be simple closed paths.

The singular points are of two general types: (1) the *line singularities* and (2) the *point singularities* consisting of the conical points and nodes. The point singularities are necessarily a set of vertices of K whose complement in S is the set of line singularities. The set of line singularities falls into components called *lines*. A line may be a simple closed path or it may be a homeomorph of the real line. In the latter case, the ends may be identified in S or they may be distinct vertices (nodes). We shall say that S is *oriented* if each of its lines is oriented. We shall henceforth suppose S to be oriented.

Let us pass now to the second barycentric subdivision of K. With this subdivision it is an elementary matter to verify the following details: the boundary of st(S) (i.e., closure st(S) $-$st(S)) consists of a finite number of disjoint simple closed paths $\{\rho_i\}$, and $K-$st(S) consists of a finite number of disjoint bounded manifolds $\{M_j\}$; the set of all the boundaries of the M_j is the sum of the ρ_i. (Of course, K may have no singular points, and so is then a closed manifold.) If a line is a 1-cell, it has a star which is a book; if a line is a simple closed path, and if we remove therefrom a vertex, the remainder of the path is a 1-cell, which we also call a line, whose star is again a book. It follows that st(S)$-S$ is a finite number of open annuli. Any such annulus has one boundary which is a ρ_i and the other boundary is mapped piecewise homeomorphically (each boundary is partitioned into a finite number of points and open 1-cells, and each such 1-cell is sent homeomorphically on a line) on a closed path of lines, or *line-path*, σ_i, in S; or it may be that some σ_i is a degenerate point-path.

Let S be oriented in a fixed way and let each orientable manifold M_j be given a fixed orientation. Let $\sigma_i(M_j)$ denote that σ_i is the image of ρ_i on M_j; we say σ_i is *incident* with M_j, $\sigma_i < M_j$, and the inverse path, σ_i^{-1}, is incident with the oppositely oriented manifold: $\sigma_i^{-1} < M_j^{-1}$.

THEOREM. *K is characterized by the following system of invariants*: (1) *the oriented singular graph S (with lines, nodes, conical points, incidences and orientations specified)*; (2) *the set of manifolds* $\{M_j\}$, *where each orientable* M_j *is given a specific orientation*; (3) *the closed line or point paths* $\{\sigma_i\}$; (4) *the correspondence* $\sigma_i(M_j)$. *The following reörientations are allowed*: (a) *reörientation of any line*; (b) *reörientation of any orientable manifold together with all of its associated paths, i.e.* M_j, $\{\sigma_i(M_j)\} \leftrightarrow M_j^{-1}$, $\{\sigma_i^{-1}(M_j^{-1})\}$; (c) *inversion of any* σ_i *incident with a nonorientable* M_j.

We must show that two complexes, K and $'K$, are homeomorphic if and only if they have isomorphic systems of invariants, that is, if, after a finite number of reörientations of the allowed sort on K or $'K$, we can give a map, f, between corresponding invariants of K and $'K$ (the invariants of $'K$ are denoted by primes) such that f is a 1-1 orientation-preserving correspondence of S with $'S$ preserving lines, nodes, conical points, incidences and orientation; f is a 1-1 correspondence $M_j \leftrightarrow 'M_j$ such that corresponding manifolds are homeomorphic and positive orientations correspond; f is a 1-1 incidence-preserving correspondence $\sigma_i \leftrightarrow '\sigma_i$, and $\sigma_i < M_j \Leftrightarrow '\sigma_i < 'M_j$, such that if σ_i is the path $g(x_1, x_2, \cdots, x_t)$ of the lines $x_1, x_2, \cdots, x_t$ in K or of the point $\sigma_i = P$, then $f(\sigma_i) = '\sigma_i = g(f(x_1), f(x_2), \cdots, f(x_t))$ or $f(P) = 'P = '\sigma_i$ respectively.

The principal problem will be to establish that the reorientations allowed, and no others, are, in fact, admissible. For then, it is virtually automatic that isomorphism implies homeomorphism, since *the system of invariants prescribes the mode of construction of a complex.* Conversely, if two complexes are homeomorphic, then they have (in the common sense of the term) isomorphic (simplicial) refinements (see [5]). But refinement does not alter the system of invariants. Hence homeomorphic complexes have isomorphic systems of invariants. Now, as to the reörientations, manifestly reörientation of a line is admissible. The other reörientations are admissible as a consequence of the following lemma.

LEMMA. *If each boundary of a bounded 2-manifold is mapped homeomorphically on itself, then the homeomorphism thus given of the total boundary on itself can always be extended to the entire manifold if the manifold is nonorientable; but if the manifold is orientable, then the homeomorphism can be extended if and only if the orientations of the various boundaries are all preserved or all reversed.*

In the proof we shall use the "cut-and-paste" polygon methods of

[1; 2; 3; 6; 7]. We shall restrict ourselves to the case of a projective plane with two boundaries and a torus with two boundaries. The projective plane with two boundaries may be considered a quotient space of a regular octagon with sides labelled and identified according to the formula $aadcd^{-1}efe^{-1}$. This may be partitioned into two polygons: $adcd^{-1}y^{-1}$ and $yefe^{-1}a$. The latter two polygons may be joined along the edge a, giving the single polygon $ydc^{-1}d^{-1}yefe^{-1}$. This is cut again into two polygons $ud^{-1}y$ and $efe^{-1}ydc^{-1}u^{-1}$. These may be joined along the edge d to give the single polygon $yyuc^{-1}u^{-1}efe^{-1}$. The latter polygonal representation of the projective plane with two boundaries prescribes an extension of the boundary homeomorphism to the whole space, reversing the orientation of one boundary and preserving that of the other.

For the torus with two boundaries, we use a decagon with the identification: $aba^{-1}b^{-1}ded^{-1}ghg^{-1}$. Let us reörient the torus: $bab^{-1}a^{-1}gh^{-1}g^{-1}de^{-1}d^{-1}$. By partitioning into two polygons, $bab^{-1}a^{-1}gh^{-1}x^{-1}$ and $xg^{-1}de^{-1}d^{-1}$ and then adjoining one polygon to the other along the common edge g we get the polygon: $bab^{-1}a^{-1}de^{-1}d^{-1}xh^{-1}x^{-1}$. The latter representation of the torus when compared with the first given is seen to provide an extension of the orientation-reversing self-homeomorphism of the entire boundary to the whole bounded torus; the orientation of the torus itself is reversed. This result shows not only the asserted possibility of extension of the boundary homeomorphism, but that the double of a bounded orientable manifold (see, e.g. [7, p. 129]) is an orientable closed manifold. Consequently, if there were a self-homeomorphism of the bounded manifold which preserved the orientation of one boundary and reversed the orientation of another boundary (and, in any case, mapping each boundary on itself) then we could extend this map to the double (using the identity on the second "half") and conclude that the double is nonorientable (it would contain a möbius band), a contradiction. q.e.d.

The problem of recognizing a general finite 2-complex can be approached component-wise. Certain components may be isolated points or linear graphs. The only significant generalization remaining is the consideration of a connected finite (simplicial) 2-complex K—it no longer being required that every edge be a face of a 2-simplex (although, there shall be at least one 2-simplex in the complex).

The varieties of neighborhoods enumerated before are now augmented by the following additional possibilities: each of the four types of neighborhoods may now be complicated by a number of 1-simplexes having the center P as vertex, but which are not faces of

2-simplexes; and, finally, P may not be a vertex of any 2-simplex, but may be vertex of 1-simplexes only, or lie in a 1-simplex which is not an edge of a 2-simplex. If P is not in the closure of some 2-simplex, then we say P lies in the *1-dimensional part* of K; the 1-dimensional part is connected to the *2-dimensional part* of the complex at a certain finite number of points of junction which we call *nodes*. But the 1-dimensional part is itself part of the singular graph S. If we include those points of the 1-dimensional part, which have neighborhoods homeomorphic to an open interval, among the line singularities; and those points of the 1-dimensional part which are not line singularities are included among the nodes, the general case and theorem may be stated as before, mutatis mutandis.

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