Bibliography

1. M. Heins, *Algebraic structure and conformal mapping*, Technical Report issued by the Institute for Advanced Study, 1957.

2. H. L. Royden, *Rings of analytic and meromorphic functions*, Trans. Amer. Math. Soc. vol. 83 (1956) pp. 269–276.

3. W. Rudin, *Some theorems on bounded analytic functions*, Trans. Amer. Math. Soc. vol. 78 (1955) pp. 333–342.

4. H. Weyl, *Die Idee der Riemannschen Flaechen*, Berlin, 1923.

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A SEMI-SIMPLE MATRIX GROUP IS OF TYPE *I*

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The purpose of the following note is to give a short simple proof of the most important special case of Harish-Chandra's result [**3**, Theorem 7] that every connected semi-simple Lie group is of type I. We shall prove that every continuous unitary representation of a connected semi-simple *matrix* group is of type I. This fact is a consequence of theorems in two papers of Godement [**1**, Theorem 2; **2**, Theorem 8]. An improved version of Godement's method is used here. The improvements are:

(1) the argument is considerably shorter;

(2) infinite dimensional nonunitary representations are not needed;

(3) the argument is completely global; no direct integrals are used.

These improvements seem not to have been noticed before.

Definition. We shall say that an algebra is of type $I_{\leq n}$ if it satisfies the identities

$$[A_1, \cdots, A_r] = \sum_{i_1, \cdots, i_r} \operatorname{sgn} \begin{pmatrix} 1 & \cdots & r \\ i_1 & \cdots & i_r \end{pmatrix} A_{i_1} A_{i_2} \cdots A_{i_r} = 0$$

for all r for which the algebra of all $n \times n$ matrices satisfies them.

In [**5**, §2], Kaplansky shows that the algebra of all $n \times n$ matrices is of type $I_{\leq n}$ but not of type $I_{\leq n-1}$. It follows from the definition that an algebra is of type $I_{\leq n}$ if it is a subalgebra of an algebra of type $I_{\leq n}$, or if it is a homomorphic image of an algebra of type $I_{\leq n}$, or if it has a separating family of homomorphisms into algebras of type $I_{\leq n}$. Since the above identities are linear in each variable, a von Neumann algebra is of type $I_{\leq n}$ if it has a weakly dense subalgebra of type $I_{\leq n}$. A von Neumann algebra of type I (in the usual sense) is

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the direct sum of von Neumann algebras of homogeneous type I_k where k runs over the cardinals. Since an algebra of type $I_{\leqq n}$ in the sense of the above definition has no subalgebras isomorphic to the $(n+1)\times(n+1)$ matrices, it is clear that a von Neumann algebra is of type $I_{\leqq n}$ as defined above if and only if it is of type I in the usual sense and all the direct summands of homogeneous type I_k are zero for $k>n$. Consequently the present terminology accords with the usual one.

THEOREM 1. *Let G be a connected semi-simple matrix group. Let L be the left regular representation of G on $L_2(G)$, and let $\mathfrak{L}$ be the von Neumann algebra generated by L. Let K be a maximal compact subgroup of G, and let ρ be an irreducible representation of K of degree n with character $\bar{\chi}/n$. Let E be the projection given by*

$$E = \int_K \chi(k)L(k)d_Kk$$

where d_Kk is Haar measure on K and the integral is in the weak sense. Then the algebra $E\mathfrak{L}E$ is of type $I_{\leqq n^2}$.

PROOF. The group G has a connected solvable subgroup S such that $G=KS$ (see [**4**, Lemma 3.11 and the proof of Lemma 3.12]). Let $\mathfrak{A}$ be the algebra of all L_ϕ with ϕ in $C_0(G)$ where $L_\phi=\int_G\phi(x)L(x)dx$. It is sufficient to show that $E\mathfrak{A}E$ is of type $I_{\leqq n^2}$. This will be done by exhibiting a separating family of homomorphisms of $E\mathfrak{A}E$ into $q\times q$ matrices with $q\leqq n^2$. Such a family is provided by the finite dimensional irreducible representations of G.

If γ is a finite dimensional representation of G we consider the homomorphism $L_f\rightarrow\int f(x)\gamma(x)dx$ of $\mathfrak{A}$ into operators on the representation space of γ. Since G has a faithful finite dimensional representation, the Stone-Weierstrass theorem implies that these homomorphisms separate $\mathfrak{A}$. Consequently, since G is semi-simple, the ones arising from irreducible γ separate $\mathfrak{A}$. Let $Q=\int_K\chi(k)\gamma(k)d_Kk$. Then $EL_fE\rightarrow Q(\int f(x)\gamma(x)dx)Q$ under the homomorphism. It is, therefore, sufficient to observe that the dimension q of the range of the projection Q is $\leqq n^2$. But this is a consequence of the fact that $\gamma|K$ is a cyclic representation, which we shall now show. By Lie's theorem there exists a vector v such that $\gamma(s)v=\lambda(s)v$ for all s in S where $\lambda(s)$ is a scalar. Since γ is irreducible $\gamma(G)v=\gamma(K)\gamma(S)v=\lambda(S)\gamma(K)v$ spans the representation space, and so $\gamma(K)v$ does also.

COROLLARY. *Let G be a connected semi-simple matrix group. Then every continuous unitary representation of G is of type I.*

PROOF. Let U be a unitary representation of G, and let ρ be any irreducible representation of a maximal compact subgroup K. Let $\bar{\chi}_\rho/n$ be the character of ρ where n is the degree of ρ. Let $P_\rho = \int \chi_\rho(x) U(x) dx$. Let $\mathfrak{U}$ be the von Neumann algebra generated by U. Then $P_\rho \mathfrak{U} P_\rho$ is of type $I_{\leqq n^2}$ since a number of subalgebras of $E\mathfrak{L}E$ have homomorphic images which are weakly dense in $P_\rho \mathfrak{U} P_\rho$. But $\{P_\rho\}$ is an orthogonal family of projections whose sum is I. Hence $\mathfrak{U}$ is of type I.

BIBLIOGRAPHY

1. R. Godement, *A theory of spherical functions* I, Trans. Amer. Math. Soc. vol. 73 (1952) pp. 496–556.

2. ———, *Théorie des caractères* II. *Définition et propriétés générales des caractères*, Ann. of Math. vol. 59 (1954) pp. 63–85.

3. Harish-Chandra, *Representations of a semi-simple Lie group on a Banach space* I, Trans. Amer. Math. Soc. vol. 75 (1953) pp. 185–243.

4. K. Iwasawa, *On some types of topological groups*, Ann. of Math. vol. 50 (1949) pp. 507–558.

5. I. Kaplansky, *Groups with representations of bounded degree*, Canad. J. Math. vol. (1949) pp. 105–112.

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