ON THE HARMONIC SUMMABILITY OF DOUBLE FOURIER SERIES

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1. Suppose that the function $f(u, v)$ is integrable in the sense of Lebesgue, over the square $(-\pi, \pi; -\pi, \pi)$ and is periodic with period 2π in each variable. Let

$$\phi(u, v) = \phi_x(u, v) = \frac{1}{4}\left[f(x+u, y+v) + f(x+u, y-v) + f(x-u, y+v) + f(x-u, y-v) - 4s\right].$$

DEFINITION. The double sequence $\{s_{m,n}\}$ is said to be summable by Harmonic means, or summable $(H, 1, 1)$ if

$$\lim_{m\to\infty, n\to\infty} \frac{1}{\log m \log n} \sum_{l=0}^{m} \sum_{k=0}^{n} \frac{s_{m-l,n-k}}{(l+1)(k+1)}$$

exists.

This is a particular case $p_n = 1/(n+1)$ of Norlund summability of a double sequence as defined by Herriot [4], Hille and Tamarkin [5] have proved the following theorem on the Harmonic Summability of Fourier Series:

THEOREM A. *The Fourier Series of the function $f(x)$ is summable $(H, 1)$ at the point x at which*

$$\phi_1(t) = \int_0^t |\phi(u)|\, du = o\left(\frac{t}{\log \frac{1}{t}}\right)$$

where $\phi(t) = f(x+t) + f(x-t) - 2f(x)$.

An easy proof of this theorem is given by Prasad and Siddiqui [6]. We shall prove the theorem:

THEOREM B. *If*

$$\Phi(u, v) = \int_0^u ds \int_0^v |\phi(s, t)|\, dt = o\left(\frac{uv}{\log \frac{1}{u} \log \frac{1}{v}}\right),$$

$$\int_0^\pi dt \left| \int_0^u \phi(s, t) ds \right| = O\left(\frac{u}{\log \frac{1}{u}}\right),$$

Received by the Editors October 3, 1957.

$$\int_0^\pi ds \left| \int_0^v \phi(s, t) dt \right| = O \left\{ \frac{v}{\log \frac{1}{v}} \right\}$$

then the double Fourier Series of function $f(u, v)$ is summable $(H, 1, 1)$ to the sum s at $u = x$ and $v = y$.

This theorem is a generalization of Theorem A for double Fourier series and also is analogous to the theorem of Chow [1] for summability $(C, 1, 1)$ of the double Fourier series.

2. We require the following lemmas:

LEMMA 1. *If $0 < t < \pi$, then*

$$\left| \sum_{k=0}^{n} \frac{\cos (k+1)t}{k+1} \right| < A \left(1 + \log \frac{1}{t} \right).$$

This is known [3].

LEMMA 2. *For all values of n and x*

$$\left| \sum_{k=0}^{n} \frac{\sin (k+1)t}{k+1} \right| \leqq \frac{1}{2} \pi + 1$$

This is known [7].

LEMMA 3. *For t such that $0 \leqq t \leqq 1/n$*

$$| k_n(t) | = \left| \frac{1}{2\pi \log n} \sum_{\gamma=0}^{n} \frac{1}{\gamma + 1} \frac{\sin (n - \gamma + 1/2)t}{\sin t/2} \right| = O(n)$$

where $k_n(t)$ is Harmonic Summability Kernel for Fourier series.

PROOF.

$$\begin{aligned} | k_n(t) | &= \left| \frac{1}{2\pi \log n} \sum_{\gamma=0}^{n} \frac{1}{\gamma + 1} \frac{\sin (n - \gamma + 1/2)t}{\sin t/2} \right| \\ &= O \left(\frac{1}{\log n} \sum_{\gamma=0}^{n} \frac{1}{\gamma + 1} \frac{(2n - 2\gamma + 1) | \sin t/2 |}{| \sin t/2 |} \right) \\ &= O \left(\frac{2n + 1}{\log n} \sum_{\gamma=0}^{n} \frac{1}{\gamma + 1} \right) \\ &= O(n). \end{aligned}$$

LEMMA 4. *For t such that $1/n \leqq t \leqq \delta$.*

$$| k_n(t) | = O\left[\frac{1}{t \log n}\{1 + \log 1/t\}\right].$$

PROOF.

$$k_n(t) = \frac{1}{2\pi \log n} \sum_{\gamma=0}^{n} \frac{1}{\gamma+1} \frac{\sin (n - \gamma + 1/2)t}{\sin t/2}$$

$$= \frac{1}{2\pi \log n \sin t/2} \sum_{\gamma=0}^{n} \frac{1}{\gamma+1}\left\{\sin\left(n + \frac{3}{2}\right)t \cos (\gamma + 1)t - \cos\left(n + \frac{3}{2}\right)t \sin (\gamma + 1)t\right\}$$

$$= \frac{1}{2\pi \log n \sin t/2}\left\{\sin\left(n + \frac{3}{2}\right)t \sum_{\gamma=0}^{n} \frac{\cos (\gamma + 1)t}{\gamma+1} - \cos\left(n + \frac{3}{2}\right)t \sum_{\gamma=0}^{n} \frac{\sin (\gamma + 1)t}{\gamma+1}\right\}.$$

$$| k_n(t) | = O\left[\frac{1}{t \log n}\left\{\left|\sum_{\gamma=0}^{n} \frac{\cos (\gamma + 1)t}{\gamma+1}\right| + \left|\sum_{\gamma=0}^{n} \frac{\sin (\gamma + 1)t}{\gamma+1}\right|\right\}\right]$$

$$= O\left[\frac{1}{t \log n}\left\{A\left(1 + \log \frac{1}{t}\right) + \frac{1}{2}\pi + 1\right\}\right],$$

by Lemma 1 and 2

$$= O\left[\frac{1}{t \log n}\left\{1 + \log \frac{1}{t}\right\}\right].$$

LEMMA 5. *For t such that* $\delta \leqq t \leqq \pi$

$$| k_n(t) | = O\left[\frac{1}{t \log n}\right].$$

PROOF. Applying Abel's transformation, we have

$$| k_n(t) | = O\left[\frac{1}{t \log n} \sum_{\gamma=0}^{n-1}\left(\frac{1}{\gamma-1} - \frac{1}{\gamma-2}\right) \cdot \frac{\sin [(n + 1/2) + v - 1/2]t \sin vt/2}{\sin t/2}\right] + O\left[\frac{1}{nt \log n}\right]$$

$$= O\left[\frac{1}{t \log n}\right].$$

3. Proof of the theorem

$$\pi^2\sigma_{m,n} = \int_0^\pi \int_0^\pi \phi(u, v) k_m(u) k_n(v) du dv$$

$$= \left(\int_0^\delta \int_0^\tau + \int_0^\delta \int_\tau^\pi + \int_\delta^\pi \int_0^\tau + \int_\delta^\pi \int_\tau^\pi \right) \phi(u, v) k_m(u) k_n(v) du dv$$

where $1/m < \delta < \pi$, $1/n < \tau < \pi$

$$= I_1 + I_2 + I_3 + I_4 \text{ say.}$$

$$| I_4 | = O\left(\frac{1}{\log m \log n} \int_\delta^\pi \int_\tau^\pi \frac{| \phi(u, v) |}{uv} du dv \right)$$

$= o(1)$ by the help of Lemma 1.

$$I_3 = \int_\delta^\pi k_m(u) du \int_0^\tau \phi(u, v) k_n(u) dv$$

$$= \int_\delta^\pi k_m(u) du \int_0^{1/n} \phi(u, v) k_n(v) dv + \int_\delta^\pi k_m(u) du \int_{1/n}^\tau \phi(u, v) k_n(u) dv$$

$$= I_{3,1} + I_{3,2} \text{ say.}$$

By Lemmas 3, 4 and the theorem,

$$| I_{3,1} | = O\left[\frac{n}{\log m} \int_0^{1/n} | \phi(u, v) | dv \right]$$

$$= o(1).$$

$$| I_{3,2} | = O\left[\frac{1}{\log m} \int_{1/n}^\tau | \phi(u, v) | \frac{1}{v \log n} \left(1 + \log \frac{1}{v} \right) dv \right]$$

$$= O\left[\frac{1}{\log m \log n} \int_{1/n}^\tau | \phi(u, v) | \frac{dv}{v} \right]$$

$$+ O\left[\frac{1}{\log m \log n} \int_{1/n}^\tau | \phi(u, v) | \frac{1}{v} \log \frac{1}{v} dv \right]$$

$$= | I_{3,2,1} | + | I_{3,2,2} | \text{ say.}$$

$$| I_{3,2,1} | = O\left[\frac{1}{\log m \log n} \left[\Phi(u, v) \frac{1}{v} \right]_{1/n}^\tau + \frac{1}{\log m \log n} \int_{1/n}^\tau \Phi(u, v) \frac{dv}{v^2} \right]$$

$$= O\left[\frac{1}{\log m \log n} \right] + o\left[\frac{1}{\log m \log n} \int_{1/n}^\tau \frac{dv}{v \log 1/v} \right]$$

$$= o(1) + o\left[\frac{\log \log 1/v}{\log n} \right]_{1/n}^\tau$$

$$= o(1).$$

$$
\begin{aligned}
|I_{3,2,2}| &= O\left[\frac{1}{\log m \log n}\int_{1/n}^{\tau} |\phi(u,v)| \frac{1}{v}\log\frac{1}{v}\,dv\right] \\
&= O\left[\frac{1}{\log m \log n}\left[\Phi(u,v)\frac{1}{v}\log\frac{1}{v}\right]_{1/n}^{\tau}\right. \\
&\qquad \left. + \frac{1}{\log m \log n}\int_{1/n}^{\tau}\Phi(u,v)\left\{1+\log\frac{1}{v}\right\}\frac{dv}{v^2}\right] \\
&= o(1) + O\left(\frac{1}{\log m \log n}\int_{1/n}^{\tau}\frac{\Phi(u,v)}{v^2}\,dv\right) \\
&\qquad + O\left(\frac{1}{\log m \log n}\int_{1/n}^{\tau}\frac{\Phi(u,v)}{v^2}\log\frac{1}{v}\,dv\right) \\
&= o\left(\frac{1}{\log n}\int_{1/n}^{\tau}\frac{dv}{v\log 1/v}\right) + o\left(\frac{1}{\log n}\int_{1/n}^{\tau}\frac{dv}{v}\right) \\
&= o\left(\left[\frac{\log\log 1/v}{\log n}\right]_{1/n}^{\tau}\right) + o\left(\left[\frac{\log 1/v}{\log n}\right]_{1/n}^{\tau}\right) \\
&= o(1).
\end{aligned}
$$

Thus $|I_3| = o(1)$. Similarly $|I_2| = o(1)$.

Let us evaluate I_1.

$$
\begin{aligned}
I_1 = \left(\int_0^{1/m}\int_0^{1/n} + \int_0^{1/m}\int_{1/n}^{\tau} + \int_{1/m}^{\delta}\int_0^{1/n} + \int_{1/m}^{\delta}\int_{1/n}^{\tau}\right) \\
\cdot\phi(u,v)k_m(u)k_n(v)\,du\,dv
\end{aligned}
$$

$$= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}, \text{ say.}$$

$$
\begin{aligned}
|I_{1,1}| &= O\left(\int_0^{1/m}\int_0^{1/n} |\phi(u,v)|\,|k_m(u)|\,|k_n(v)|\,du\,dv\right) \\
&= O\left(\int_0^{1/m}\int_0^{1/n} |\phi(u,v)|\, mn\,du\,dv\right) \\
&= o(1).
\end{aligned}
$$

$$
\begin{aligned}
|I_{1,2}| &= O\left[\int_0^{1/m} m\,du \int_{1/n}^{\tau}\phi(u,v)k_n(v)\,dv\right] \\
&= O\left[\int_{1/n}^{\tau}\phi(u,v)k_n(v)\,dv\right] \\
&= o(1) \text{ as } I_{3,2}.
\end{aligned}
$$

Thus $|I_{1,2}| = o(1)$. Similarly $|I_{1,3}| = o(1)$.

$$I_{1,4} = \int_{1/m}^{\delta}\int_{1/n}^{\tau} \phi(u, v) k_m(u) k_n(v)\, du dv.$$

$$|I_{1,4}| = O\Big[\int_{1/m}^{\delta}\int_{1/n}^{\tau} |\phi(u, v)| \frac{1}{u \log m}\Big\{1 + \log\frac{1}{u}\Big\}\frac{1}{v \log n} \cdot \Big\{1 + \log\frac{1}{v}\Big\} dudv\Big]$$

$$= O\Big[\int_{1/m}^{\delta}\int_{1/n}^{\tau} |\phi(u, v)| \frac{1}{u \log m}\cdot\frac{1}{v \log n} dudv\Big]$$

$$+ O\Big[\int_{1/m}^{\delta}\int_{1/n}^{\tau} |\phi(u, v)| \frac{1}{u \log m}\log\frac{1}{u}\frac{1}{v \log n} dudv\Big]$$

$$+ O\Big[\int_{1/m}^{\delta}\int_{1/n}^{\tau} |\phi(u, v)| \frac{1}{u \log m}\frac{1}{v \log n}\log\frac{1}{v} dudv\Big]$$

$$+ O\Big[\int_{1/m}^{\delta}\int_{1/n}^{\tau} |\phi(u, v)| \frac{1}{u \log m}\log\frac{1}{u}\frac{1}{v \log n}\log\frac{1}{v} dudv\Big]$$

$$= |I_{1,4,1}| + |I_{1,4,2}| + |I_{1,4,3}| + |I_{1,4,4}|.$$

$$|I_{1,4,1}| = O\Big[\int_{1/m}^{\delta}\frac{1}{u \log m} du \int_{1/n}^{\tau}\frac{|\phi(u, v)|}{v \log n} dv\Big]$$

$$= O\Big[\frac{1}{\log n}\int_{1/n}^{\tau}\frac{|\phi(u, v)|}{v} dv\Big]$$

$= o(1)$ by the help of $I_{3,2,1}$.

$$|I_{1,4,2}| = O\Big[\int_{1/n}^{\tau}\frac{1}{v \log n} dv \int_{1/m}^{\delta} |\phi(u, v)| \frac{1}{u \log m}\log\frac{1}{u} du\Big]$$

$$= O\Big[\frac{1}{\log m}\int_{1/m}^{\delta} |\phi(u, v)| \frac{1}{u}\log\frac{1}{u} du\Big]$$

$= o(1)$ by $I_{3,2,2}$.

$|I_{1,4,3}| = o(1)$ as $|I_{1,4,2}|$.

$$|I_{1,4,4}| = O\Big[\int_{1/m}^{\delta}\int_{1/n}^{\tau} |\phi(u, v)| \frac{1}{u \log m}\cdot\log\frac{1}{u}\frac{1}{v \log n}\log\frac{1}{v} dudv\Big].$$

By partial integration for double integral we have [2],

$$\Phi(\delta, \tau) \frac{1}{\delta \log m} \log \frac{1}{\delta} \frac{1}{\delta \log n} \log \frac{1}{\delta}$$

$$- \frac{1}{\tau \log n} \log \frac{1}{\tau} \int_{1/m}^{\delta} \Phi(u, \tau) \frac{1 - \log 1/u}{u^2 \log m} du$$

$$- \frac{1}{\delta \log m} \log \frac{1}{\delta} \int_{1/n}^{\tau} \Phi(\delta, v) \frac{1 - \log 1/v}{v^2 \log n} dv$$

$$+ \int_{1/m}^{\delta} \int_{1/n}^{\tau} \Phi(u, v) \frac{(1 - \log 1/u)(1 - \log 1/v)}{u^2 v^2 \log m \log n} dudv$$

$$= L_1 + L_2 + L_3 + L_4.$$

$$L_1 = o(1).$$

$$L_2 = o\left(\int_{1/m}^{\delta} \Phi(u, \tau) \frac{1}{u^2 \log m} du\right) + o\left(\int_{1/n}^{\tau} \Phi(u, \tau) \frac{\log 1/u}{u^2} du\right)$$

$= o(1)$ by $I_{3,2,1}$ and $I_{3,2,2}$.

$L_3 = o(1)$ as in L_2.

$$L_4 = \int_{1/m}^{\delta} \int_{1/n}^{\tau} \frac{\Phi(u, v) dudv}{u^2 v^2 \log m \log n}$$

$$- \int_{1/m}^{\delta} \int_{1/n}^{\tau} \frac{\Phi(u, v) \log (1/u) dudv}{u^2 v^2 \log m \log n}$$

$$- \int_{1/m}^{\delta} \int_{1/n}^{\tau} \frac{\Phi(u, v) \log (1/v) dudv}{u^2 v^2 \log m \log n}$$

$$+ \int_{1/m}^{\delta} \int_{1/n}^{\tau} \frac{\Phi(u, v) \log (1/u) \log (1/v) dudv}{u^2 v^2 \log m \log n}$$

$$= L_{4,1} + L_{4,2} + L_{4,3} + L_{4,4}.$$

$$L_{4,1} = o\left(\int_{1/m}^{\delta} \frac{du}{u \log 1/u} \int_{1/n}^{\tau} \frac{dv}{v \log 1/v}\right)$$

$$= o(1).$$

$$L_{4,2} = \left(\int_{1/m}^{\delta} \int_{1/n}^{\tau} \frac{1}{u} \frac{1}{v \log 1/v} \cdot \frac{1}{\log m \log n} dudv\right)$$

$$= o\left(\int_{1/m}^{\delta} \frac{du}{u \log m} \int_{1/n}^{\tau} \frac{du}{v \log 1/v}\right)$$

$$= o(1).$$

Similarly $L_{4,3}=o(1)$.

$$L_{4,4} = o\left(\int_{1/m}^{\delta} \frac{du}{u \log m} \int_{1/n}^{\tau} \frac{dv}{v \log n}\right.$$
$$= o(1).$$

Thus the proof is complete.

I am much indebted to Dr. M. L. Misra for his kind help and guidance in the preparation of this paper.

References

1. Y. S. Chow, *On the cesaro summability of double Fourier series*, Tôhoku Math. J. vol. 5 (1953) pp. 277–283.

2. J. J. Gergen, *Convergence criteria for double Fourier series*, Trans. Amer. Math. Soc. vol. 35 (1933) pp. 29–63.

3. Hardy and Rogosinki, Proc. Cambridge Philos. Soc. vol. 43 (1947) pp. 10–25.

4. J. G. Herriot, *The Nörlund summability of double Fourier series*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 72–94.

5. Hille and Tamarkin, *On the summability of Fourier series*, Trans. Amer. Math. Soc. vol. 34 (1932) pp. 757–783.

6. B. N. Prasad and Siddiqui, *Harmonic summability of Fourier series*, Proc. Indian Acad. Sci. vol. 28 (1948) pp. 527–531.

7. E. C. Titchmarsh, *Theory of functions*, p. 40.

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