ON COMPLETELY CONTINUOUS HANKEL MATRICES

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1. Main theorem. Let \( x = (x_0, x_1, \ldots) \) be a sequence of complex numbers and let \( |x| \) be the norm \( |x| = \left( \sum |x_n|^2 \right)^{1/2} \geq 0 \). Let an asterisk denote complex conjugation. If \( |x| < \infty \) and \( |y| < \infty \), let \([x, y] = [y, x]\) denote the sum of the series \( \sum x_n y_n \), that is, the "scalar" product of \( y \) by \( x^* \).

For a given \( x \) with a finite norm \( |x| \), let \( x(t) \) denote a function of class \( L^2(0, 2\pi) \) having the Fourier series

\[
(1) \quad x(t) \sim \sum_{n=0}^{\infty} x_n e^{i n t}.
\]

Functions which differ only on zero sets will be considered to be identical. Correspondingly, \( x(t) \) will be called continuous if there exists a continuous periodic function \( x(t) \) satisfying \((1)\). Also, "bounded" and "sup" below mean "essentially bounded" and "ess sup."

With an \( f = (f_0, f_1, \ldots) \) of finite norm, associate the infinite Hankel matrix

\[
(2) \quad H = H(f) = (f_{n+m}) \text{, where } n, m = 0, 1, \ldots.
\]

Then \([Hx, y]\) is the bilinear form \( \sum \sum f_{n+m} x_n y_m \). If \( H \) is bounded in the sense of Hilbert, put

\[
(3) \quad \|H\| = \text{l.u.b.} \quad |Hx| = \text{l.u.b.} \quad |[Hx, y]|.
\]

Although it is not assumed that \( H \) is hermitian (that is, that the components of \( f \) are real-valued), it is easy to see that

\[
\|H\| = \text{l.u.b.} \quad |Hx, x| \text{ for } |x| = 1.
\]

It was pointed out by Toeplitz \([6]\) that a sufficient condition for \( H(f) \) to be bounded is that one of the two functions \( F_1(t) \sim f_0/2 + \sum f_n \cos nt \), \( F_2(t) \sim i \sum f_{n-1} \sin nt \) be bounded on \((0, 2\pi)\). Recently, Nehari \([5]\) gave the following extension of Toeplitz's sufficient condition to a necessary and sufficient condition for the boundedness of \( H(f) \):

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(i) (Nehari) Let \( f = (f_0, f_1, \cdots) \) have a finite norm. If 
\( g = (0, g_1, g_2, \cdots) \) is of finite norm and

\[
F(t) = f(-t) + g(t) \sim \sum_{n=0}^{\infty} f_n e^{-int} + \sum_{n=1}^{\infty} g_n e^{int},
\]

then \( \|H(f)\| \leq \sup |F(t)| (\leq \infty) \). \( H(f) \) is bounded if and only if there exists a \( g = (0, g_1, \cdots) \) of finite norm such that (4) is bounded; in which case, there exists a \( g \) with the property that

\[
\|H(f)\| = \sup |F(t)|.
\]

It was pointed out in [4] that a sufficient condition for \( H(f) \) to be completely continuous is that one of the two functions \( F_1(t), F_2(t) \), above, be a continuous, periodic function. The main result of this paper is the following analogue of (i):

(ii) \( H(f) \) is completely continuous if and only if there exists a \( g = (0, g_1, \cdots) \) of finite norm such that (4) is continuous.

2. Remarks. When \( H = H(f) \) is completely continuous or whenever \( H(f) \) has the property that \( \|[Hx, x]\| = \|H\| \) for some \( x = x^0 \) of unit norm, then a function (4) satisfying (5) also satisfies

\[
|F(t)| \equiv \text{const.} (=\|H\|)
\]

and \( F(t)x(t) = \|H\|x^*(t) \) for \( x = x^0 \). This follows readily from the Toeplitz relation

\[
(2\pi)^{-1} \int_0^{2\pi} F(t)x^2(t)dt = [Hx, x]
\]

for any (bounded) function (4) and any \( x \) (of finite norm) and from the theorem of F. and M. Riesz which states that if \( x(t) \neq 0 \), then \( x(t) \) vanishes only on a zero set.

Furthermore, in this case, a function (4) satisfying (5), hence (6), is unique. For if \( F^j(t) = f(-t) + g^j(t) \), where \( j = 1, 2 \), satisfy (4)–(5), then the same is true of \( F(t) = F^1(t) + \lambda(g^2(t) - g^1(t)) \) for \( 0 \leq \lambda \leq 1 \). But \( F(t) \) satisfies (6) for \( 0 \leq \lambda \leq 1 \), so that \( g^2(t) - g^1(t) \equiv 0 \).

In the case of a completely continuous \( H(f) \), it will remain undecided whether or not the unique \( F(t) \) satisfying (4)–(6) is a continuous periodic function.

Nehari’s proof of (i) depends on the following theorem of Carathéodory and Fejér [1]: Among all the functions \( G(z) = a_0 + a_1z + \cdots \), analytic and bounded on \( |z| < 1 \) with the same \( n + 1 \) initial coefficients \( a_0, \cdots, a_n \), there exists a unique one for which

\[
M = \text{l.u.b.} |G(z)| \text{ on } |z| < 1
\]
is the smallest; this unique \( G(z) \) is of the form

\[ G(z) = Me^{i\phi} \prod_{k=1}^{n} \frac{\alpha_k - z}{1 - \alpha_k^*z}, \]

where \( \phi \) is a real constant and \( \alpha_1, \ldots, \alpha_n \) are complex numbers satisfying \( |\alpha_k| < 1 \). In particular, \( G(z) \) is analytic on \( |z| \leq 1 \) and \( |G(e^{i\theta})| = M \).

Note that if \( f = (f_0, f_1, \ldots) \) has only a finite number of components distinct from 0, say \( f_j = 0 \) if \( j > n \), then the unique \( F(t) \) satisfying (4)—(6) is given by \( F(t) = e^{-int}G(e^{it}) \), where \( G(z) \) is the function belonging to the \( n+1 \) constants \((a_0, \ldots, a_n) = (f_n, f_{n-1}, \ldots, f_0)\).


3. Two lemmas. The proof of (ii) will depend on the remarks of the last section and on either of the following two equivalent lemmas.

**Lemma 1.** Let \( f = (f_0, f_1, \ldots) \) have a finite norm. \( H(f) \) is bounded [completely continuous] if and only if there exists a sequence of vectors \( f^n = (f_0^n, f_1^n, \ldots) \) each having only a finite number of nonvanishing components and such that \( H_n = H(f^n) \) tends strongly [uniformly] to \( H \) as \( n \to \infty \).

In order to be able to state the second lemma, introduce the following notation: For a given vector \( f = (f_0, f_1, \ldots) \) and number \( r, 0 < r < 1 \), let \( f^r = (f_0, rf_1, r^2f_2, \ldots) \) and \( f^{r\cdot n} = (f_0, rf_1, \ldots, r^nf_n, 0, \ldots) \).

**Lemma 2.** Let \( f = (f_0, f_1, \ldots) \) have a finite norm. \( H(f) \) is bounded [completely continuous] if and only if \( H_r = H(f^r) \) tends strongly [uniformly] to \( H(f) \) as \( r \to 1 - 0 \).

4. Proof of the lemmas. Note that \( H_n \) in Lemma 1 is completely continuous, in particular, bounded. Consequently, a strong [uniform] limit of \( H_1, H_2, \ldots \) is bounded [completely continuous]. This implies the "if" half of Lemma 1.

Note that for any pair of vectors \( f^1 \) and \( f^2 \), one has, by (i) and by \( H(f^1 - f^2) = H(f^1) - H(f^2) \),

\[ \|H(f^1) - H(f^2)\| \leq \sup |f^1(t) - f^2(t)| \leq \infty. \]

(9)

Also, if \( r \) is fixed on \( 0 < r < 1 \), then

\[ \sup |f^r(t) - f^{r\cdot n}(t)| \to 0 \quad \text{as} \ n \to \infty. \]

(10)

Thus \( H(f^{r\cdot n}) \) tends uniformly to \( H(f^r) \) as \( n \to \infty \).
By the half of the Lemma 1 already verified, it follows that $H(f^r)$ is completely continuous for $0 < r < 1$. Correspondingly, the “if” half of Lemma 2 follows.

The “only if” part of Lemma 1 follows from (9), (10) and from the “only if” part of Lemma 2. Thus it only remains to prove the “only if” part of Lemma 2.

If $H = H(f)$ is bounded and $x$ is any vector of finite norm, then

$$
| (H - H_r)x |^2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_{n+m}x_m - r^n \sum_{m=0}^{\infty} f_{n+m}r^mx_m |^2.
$$

Thus $| (H - H_r)x |$ does not exceed

$$
| H(x - x^r) | + \left\{ \sum_{n=0}^{\infty} (1 - r^n)^2 \sum_{m=0}^{\infty} f_{n+m}r^mx_m \right\}^{1/2}.
$$

Consequently,

$$
(11) \quad | (H - H_r)x | \leq | H(x - x^r) | + (1 - r) | Hx^r |.
$$

Since $| x^r - x | \to 0$ as $r \to 1$, the boundedness of $H$ implies that $| H(x - x^r) | \to 0$ as $r \to 1$. Also, $(1 - r) | Hx^r | \leq (1 - r) \| H \| \| x | \to 0$ as $r \to 1$. Thus $H_r$ tends strongly to $H$ as $r \to 1 - 0$.

Suppose that $H = H(f)$ is completely continuous and suppose, if possible, that $H_r$ does not tend uniformly to $H$ as $r \to 1$. Then there exists a constant $c > 0$, a sequence of $r$-values $r_1 < r_2 < \cdots$ and a sequence of unit vectors $x_1, x_2, \cdots$ such that $r_n \to 1$ as $n \to \infty$ and

$$
(12) \quad | (H - H_r) x_n | \geq c > 0 \quad \text{for} \quad r = r_n.
$$

It can be supposed that $x = \lim x_n$, as $n \to \infty$, exists weakly. Then

$$
(13) \quad y_n \equiv x_n - x \to 0 \quad \text{weakly as} \quad n \to \infty.
$$

By the triangular inequality, $| (H - H_r)x_n | \leq | (H - H_r)y_n | + | (H - H_r)x |$. As shown above, $| (H - H_r)x | \to 0$ as $r \to 1$. Thus (12) leads to a contradiction if it is shown that

$$
(14) \quad | (H - H_r)y_n | \to 0 \quad \text{as} \quad r = r_n \to 1.
$$

The inequality (11), with $x$ replaced by $y_n$, gives

$$
| (H - H_r)y_n | \leq | Hy_n | + | H(y_n)^r | + (1 - r) | H(y_n)^r |,
$$

where $(y_n)^r = (y_n^0, y_n^1, \cdots)$ if $y_n = (y_0^n, y_1^n, \cdots)$. It is easy to see that (13) implies that $(y_n)^r \to 0$ weakly as $r = r_n \to 1$. Thus, from the complete continuity of $H$, it follows that $| Hy_n | \to 0$ and $| H(y_n)^r | \to 0$ if $r = r_n$ and $n \to \infty$. This gives (14) and proves the two lemmas.
5. Proof of (ii). The "if" part of (ii) is simple and is proved as is (b) in [4, p. 365]. In fact, if there exists a \( g = (0, g_1, \cdots) \) such that \( F(t) \) in (4) is continuous, then there exist trigonometric polynomials \( F_n(t) = f^n(-t) + g^n(t) \) which tend uniformly to \( F(t) \) as \( n \to \infty \). Since \( H(f - f^n) = H(f) - H(f^n) \) and \( \|H(f - f^n)\| \leq \max |F(t) - F_n(t)| \to 0 \) as \( n \to \infty \), it follows from Lemma 1 that \( H \) is completely continuous.

Conversely, suppose that \( H = H(f) \) is completely continuous. Then, by Lemma 1, there exist vectors \( f_1, f_2, \cdots \), each having only a finite number of nonvanishing components and satisfying \( \|H - H(f^n)\| \to 0 \) as \( n \to \infty \). On replacing the sequence by a subsequence, if necessary, it can be supposed that

\[
\|H_n - H_{n+1}\| < 2^{-n}, \quad \text{where} \quad H_n = H(f^n),
\]

for \( n = 1, 2, \cdots \).

Since \( f^n - f^{n+1} \) has only a finite number of nonvanishing components, there exists a unique vector \( g^n = (0, g^n_1, \cdots) \) such that \( g^n(t) \) is continuous and

\[
|f^{n+1}(-t) - f^n(-t) + g^n(t)| \equiv \|H_n - H_{n+1}\|;
\]

cf. §2. By (15) and (16), it follows that

\[
F(t) = \lim_{n \to \infty} \left[ f^{n+1}(-t) + \sum_{k=1}^{n} g^k(t) \right]
\]

exists uniformly for all \( t \) and, hence, is a continuous, periodic function.

It is clear that \( |f| \leq \|H\|. \) In fact, the first component of \( Hf^* \) is \( |f|^2 \), so that \( |f|^2 \leq \|Hf^*\| \leq \|H\||f| \). Hence, \( \|H - H_n\| \to 0 \) implies that \( |f - f^n| \to 0 \) as \( n \to \infty \). This shows that (17) is of the type (4) and completes the proof of (ii).

References


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