ON COMPLETELY CONTINUOUS HANKEL MATRICES[1]

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1. **Main theorem.** Let $x=(x_0, x_1, \cdots)$ be a sequence of complex numbers and let $|x|$ be the norm $|x|=(\sum|x_n|^2)^{1/2}\geqq 0$. Let an asterisk denote complex conjugation. If $|x|<\infty$ and $|y|<\infty$, let $[x, y] = [y, x]$ denote the sum of the series $\sum x_n y_n$, that is, the "scalar" product of y by x^*.

For a given x with a finite norm $|x|$, let $x(t)$ denote a function of class $L^2(0, 2\pi)$ having the Fourier series

$$x(t) \sim \sum_{n=0}^{\infty} x_n e^{int}. \tag{1}$$

Functions which differ only on zero sets will be considered to be identical. Correspondingly, $x(t)$ will be called continuous if there exists a continuous periodic function $x(t)$ satisfying (1). Also, "bounded" and "sup" below mean "essentially bounded" and "ess sup."

With an $f=(f_0, f_1, \cdots)$ of finite norm, associate the infinite Hankel matrix

$$H = H(f) \equiv (f_{n+m}), \quad \text{where} \quad n, m = 0, 1, \cdots. \tag{2}$$

Then $[Hx, y]$ is the bilinear form $\sum\sum f_{n+m}x_n y_m$. If H is bounded in the sense of Hilbert, put

$$\|H\| = \underset{|x|=1}{\text{l.u.b.}} \ |Hx| = \underset{|x|=|y|=1}{\text{l.u.b.}} \ |[Hx, y]|. \tag{3}$$

Although it is not assumed that H is hermitian (that is, that the components of f are real-valued), it is easy to see that

$$\|H\| = \text{l.u.b.} |[Hx, x]| \text{ for } |x|=1.$$

It was pointed out by Toeplitz [6] that a sufficient condition for $H(f)$ to be bounded is that one of the two functions $F_1(t)\sim f_0/2 + \sum f_n \cos nt$, $F_2(t)\sim i\sum f_{n-1} \sin nt$ be bounded on $(0, 2\pi)$. Recently, Nehari [5] gave the following extension of Toeplitz's sufficient condition to a necessary and sufficient condition for the boundedness of $H(f)$:

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(i) (Nehari) *Let* $f=(f_0, f_1, \cdots)$ *have a finite norm. If* $g=(0, g_1, g_2, \cdots)$ *is of finite norm and*

$$F(t) = f(-t) + g(t) \sim \sum_{n=0}^{\infty} f_n e^{-int} + \sum_{n=1}^{\infty} g_n e^{int}, \tag{4}$$

then $\|H(f)\| \leqq \sup |F(t)| (\leqq \infty)$. $H(f)$ *is bounded if and only if there exists a* $g=(0, g_1, \cdots)$ *of finite norm such that* (4) *is bounded*; *in which case, there exists a* g *with the property that*

$$\|H(f)\| = \sup |F(t)|. \tag{5}$$

It was pointed out in [4] that a sufficient condition for $H(f)$ to be completely continuous is that one of the two functions $F_1(t)$, $F_2(t)$, above, be a continuous, periodic function. The main result of this paper is the following analogue of (i):

(ii) $H(f)$ *is completely continuous if and only if there exists a* $g=(0, g_1, \cdots)$ *of finite norm such that* (4) *is continuous.*

2. **Remarks.** When $H=H(f)$ is completely continuous or whenever $H(f)$ has the property that $|[Hx, x]| = \|H\|$ for some $x=x^0$ of unit norm, then a function (4) satisfying (5) also satisfies

$$|F(t)| \equiv \text{const.} (= \|H\|) \tag{6}$$

and $F(t)x(t) = \|H\| x^*(t)$ for $x=x^0$. This follows readily from the Toeplitz relation

$$(2\pi)^{-1} \int_0^{2\pi} F(t)x^2(t)dt = [Hx, x] \tag{7}$$

for any (bounded) function (4) and any x (of finite norm) and from the theorem of F. and M. Riesz which states that if $x(t) \not\equiv 0$, then $x(t)$ vanishes only on a zero set.

Furthermore, in this case, a function (4) satisfying (5), hence (6), is unique. For if $F^j(t) = f(-t) + g^j(t)$, where $j=1, 2$, satisfy (4)–(5), then the same is true of $F(t) = F^1(t) + \lambda(g^2(t) - g^1(t))$ for $0 \leqq \lambda \leqq 1$. But $F(t)$ satisfies (6) for $0 \leqq \lambda \leqq 1$, so that $g^2(t) - g^1(t) \equiv 0$.

In the case of a completely continuous $H(f)$, it will remain undecided whether or not the unique $F(t)$ satisfying (4)–(6) is a continuous periodic function.

Nehari's proof of (i) depends on the following theorem of Carathéodory and Fejér [1]: Among all the functions $G(z) = a_0 + a_1 z + \cdots$, analytic and bounded on $|z| < 1$ with the same $n+1$ initial coefficients $a_0, \cdots, a_n$, there exists a unique one for which

$$M = \text{l.u.b. } |G(z)| \text{ on } |z| < 1$$

is the smallest; this unique $G(z)$ is of the form

$$G(z) = Me^{i\phi} \prod_{k=1}^{n} (\alpha_k - z)/(1 - \alpha_k^* z), \tag{8}$$

where ϕ is a real constant and $\alpha_1, \cdots, \alpha_n$ are complex numbers satisfying $|\alpha_k| < 1$. In particular, $G(z)$ is analytic on $|z| \leqq 1$ and $|G(e^{it})| \equiv M$.

Note that if $f = (f_0, f_1, \cdots)$ has only a finite number of components distinct from 0, say $f_j = 0$ if $j > n$, then the unique $F(t)$ satisfying (4)–(6) is given by $F(t) = e^{-int}G(e^{it})$, where $G(z)$ is the function belonging to the $n+1$ constants $(a_0, \cdots, a_n) = (f_n, f_{n-1}, \cdots, f_0)$.

A short proof of the Carathéodory-Fejér theorem is given by Nehari [5]; cf. Gronwall [3]. Curiously, Fischer [2] gives a proof of a related theorem of Carathéodory using finite sections of a Hankel matrix.

3. **Two lemmas.** The proof of (ii) will depend on the remarks of the last section and on either of the following two equivalent lemmas.

LEMMA 1. *Let $f = (f_0, f_1, \cdots)$ have a finite norm. $H(f)$ is bounded [completely continuous] if and only if there exists a sequence of vectors $f^n = (f_0^n, f_1^n, \cdots)$ each having only a finite number of nonvanishing components and such that $H_n = H(f^n)$ tends strongly [uniformly] to H as $n \to \infty$.*

In order to be able to state the second lemma, introduce the following notation: For a given vector $f = (f_0, f_1, \cdots)$ and number r, $0 < r < 1$, let $f^r = (f_0, rf_1, r^2 f_2, \cdots)$ and $f^{r,n} = (f_0, rf_1, \cdots, r^n f_n, 0, \cdots)$.

LEMMA 2. *Let $f = (f_0, f_1, \cdots)$ have a finite norm. $H(f)$ is bounded [completely continuous] if and only if $H_r = H(f^r)$ tends strongly [uniformly] to $H(f)$ as $r \to 1-0$.*

4. **Proof of the lemmas.** Note that H_n in Lemma 1 is completely continuous, in particular, bounded. Consequently, a strong [uniform] limit of $H_1, H_2, \cdots$ is bounded [completely continuous]. This implies the "if" half of Lemma 1.

Note that for any pair of vectors f^1 and f^2, one has, by (i) and by $H(f^1 - f^2) = H(f^1) - H(f^2)$,

$$\|H(f^1) - H(f^2)\| \leqq \sup |f^1(t) - f^2(t)| \leqq \infty. \tag{9}$$

Also, if r is fixed on $0 < r < 1$, then

$$\sup |f^r(t) - f^{r,n}(t)| \to 0 \qquad \text{as } n \to \infty. \tag{10}$$

Thus $H(f^{r,n})$ tends uniformly to $H(f^r)$ as $n \to \infty$.

By the half of the Lemma 1 already verified, it follows that $H(f^r)$ is completely continuous for $0<r<1$. Correspondingly, the "if" half of Lemma 2 follows.

The "only if" part of Lemma 1 follows from (9), (10) and from the "only if" part of Lemma 2. Thus it only remains to prove the "only if" part of Lemma 2.

If $H=H(f)$ is bounded and x is any vector of finite norm, then

$$\left| (H - H_r)x \right|^2 = \sum_{n=0}^{\infty} \left| \sum_{m=0}^{\infty} f_{n+m}x_m - r^n \sum_{m=0}^{\infty} f_{n+m}r^m x_m \right|^2 .$$

Thus $|(H-H_r)x|$ does not exceed

$$\left| H(x - x^r) \right| + \left\{ \sum_{n=0}^{\infty} (1 - r^n)^2 \left| \sum_{m=0}^{\infty} f_{n+m}r^m x_m \right|^2 \right\}^{1/2} .$$

Consequently,

$$\left| (H - H_r)x \right| \leqq \left| H(x - x^r) \right| + (1 - r) \left| Hx^r \right| . \tag{11}$$

Since $|x^r-x|\to 0$ as $r\to 1$, the boundedness of H implies that $|H(x-x^r)|\to 0$ as $r\to 1$. Also, $(1-r)|Hx^r|\leqq(1-r)\|H\||x|\to 0$ as $r\to 1$. Thus H_r tends strongly to H as $r\to 1-0$.

Suppose that $H=H(f)$ is completely continuous and suppose, if possible, that H_r does not tend uniformly to H as $r\to 1$. Then there exists a constant $c>0$, a sequence of r-values $r_1<r_2<\cdots$ and a sequence of unit vectors $x^1, x^2, \cdots$ such that $r_n\to 1$ as $n\to\infty$ and

$$\left| (H - H_r)\, x^m \right| \geqq c > 0 \qquad \text{for } r = r_n. \tag{12}$$

It can be supposed that $x=\lim x^n$, as $n\to\infty$, exists weakly. Then

$$y^n \equiv x^n - x \to 0 \quad \text{weakly} \quad \text{as } n \to \infty. \tag{13}$$

By the triangular inequality, $|(H-H_r)x^n| \leqq |(H-H_r)y^n| + |(H-H_r)x|$. As shown above, $|(H-H_r)x|\to 0$ as $r\to 1$. Thus (12) leads to a contradiction if it is shown that

$$\left| (H - H_r)y^n \right| \to 0 \quad \text{as} \quad r = r_n \to 1. \tag{14}$$

The inequality (11), with x replaced by y^n, gives

$$\left| (H - H_r)y^n \right| \leqq \left| Hy^n \right| + \left| H(y^n)^r \right| + (1 - r) \left| H(y^n)^r \right| ,$$

where $(y^n)^r=(y_0^n, ry_1^n, \cdots)$ if $y^n=(y_0^n, y_1^n, \cdots)$. It is easy to see that (13) implies that $(y^n)^r\to 0$ weakly as $r=r_n\to 1$. Thus, from the complete continuity of H, it follows that $|Hy^n|\to 0$ and $|H(y^n)^r|\to 0$ if $r=r_n$ and $n\to\infty$. This gives (14) and proves the two lemmas.

5. **Proof of** (ii). The "if" part of (ii) is simple and is proved as is (b) in [**4**, p. 365]. In fact, if there exists a $g=(0, g_1, \cdots)$ such that $F(t)$ in (4) is continuous, then there exist trigonometric polynomials $F^n(t)=f^n(-t)+g^n(t)$ which tend uniformly to $F(t)$ as $n\to\infty$. Since $H(f-f^n)=H(f)-H(f^n)$ and $\|H(f-f^n)\|\leqq\max|F(t)-F^n(t)|\to 0$ as $n\to\infty$, it follows from Lemma 1 that H is completely continuous.

Conversely, suppose that $H=H(f)$ is completely continuous. Then, by Lemma 1, there exist vectors $f^1, f^2, \cdots$, each having only a finite number of nonvanishing components and satisfying $\|H-H(f^n)\|\to 0$ as $n\to\infty$. On replacing the sequence by a subsequence, if necessary, it can be supposed that

$$(15) \qquad \|H_n - H_{n+1}\| < 2^{-n}, \quad \text{where} \quad H_n = H(f^n),$$

for $n=1, 2, \cdots$.

Since f^n-f^{n+1} has only a finite number of nonvanishing components, there exists a unique vector $g^n=(0, g_1^n, \cdots)$ such that $g^n(t)$ is continuous and

$$(16) \qquad |f^{n+1}(-t) - f^n(-t) + g^n(t)| \equiv \|H_n - H_{n+1}\|;$$

cf. §2. By (15) and (16), it follows that

$$(17) \qquad F(t) = \lim_{n\to\infty}\left[f^{n+1}(-t) + \sum_{k=1}^{n} g^k(t)\right]$$

exists uniformly for all t and, hence, is a continuous, periodic function.

It is clear that $|f|\leqq\|H\|$. In fact, the first component of Hf^* is $|f|^2$, so that $|f|^2\leqq|Hf^*|\leqq\|H\|\,|f|$. Hence, $\|H-H_n\|\to 0$ implies that $|f-f^n|\to 0$ as $n\to\infty$. This shows that (17) is of the type (4) and completes the proof of (ii).

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