APPROXIMATION BY A POLYNOMIAL AND ITS DERIVATIVES ON CERTAIN CLOSED SETS

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The work on the theory of approximations initiated by Weierstrass and continued by Walsh, Keldysh, and Lavrentiev, among others, has culminated in the following theorem of Mergelyan (See Mergelyan [**3**]): Given any compact subset C of the complex plane, which does not separate the plane, and given any continuous function f on C which is analytic interior to C, then f can be approximated uniformly on C by polynomials.

This theorem leaves the following question unanswered: If $f_0, f_1, \cdots, f_n$ are continuous functions on C, can a sequence $\{p_k\}$ of polynomials be found with the property that for each integer i with $0 \leqq i \leqq n$ the sequence $\{p_k^{[i]}\}$, where $p_k^{[i]}$ denotes the ith derivative of p_k, converges uniformly on C to f_i? If C is totally disconnected, it is easy to show that the answer to this question is always yes. We omit the simple proof, because a more general result will be given elsewhere. If C is a Jordan arc, the question becomes more complicated. It is clear that if C has a rectifiable sub-arc J, whose endpoints we call z_0 and z_1, then for the approximation to be possible it is necessary that $\int_J f_{i+1}(z)dz = f_i(z_1) - f_i(z_0)$ for $0 \leqq i \leqq n-1$. Thus, if the approximation is to be possible whatever the functions $f_0, f_1, \cdots, f_n$, it is necessary that C have no rectifiable sub-arcs. Conversely, if C is a Jordan arc having no rectifiable sub-arcs, we conjecture that the approximation is always possible. It is the purpose of this paper to prove this conjecture by means of an additional hypothesis, that C satisfy a Lipschitz condition of a fixed order c at a dense set of points. (This concept will be defined below.) The author has been unable to prove the conjecture without this restriction.

If S_1 and S_2 are any subsets of the complex plane, define $d(S_1, S_2) = \min \{ |z_1 - z_2| \mid z_1 \in S_1, z_2 \in S_2 \}$.

Definition 1. Let ϕ be a homeomorphic map of $[0, 1]$ into the complex plane, so that $\phi[0, 1]$ is a Jordan arc C. We say that C satisfies a Lipschitz condition of order c at a point $\phi(t)$ of C, $t \in [0, 1]$, if there exist $A > 0$ and $\delta > 0$ such that $\max \{ d(\phi[0, t], z), d(\phi[t, 1], z) \} \geqq A |\phi(t) - z|^c$ whenever $|\phi(t) - z| < \delta$.

Then we have

Theorem 1. *If C has no rectifiable sub-arcs and if there exists $c > 0$ such that C satisfies a Lipschitz condition of order c at a dense set S of*

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points, then for any continuous functions $f_0, \cdots, f_n$ *on* C *there exists a sequence* $\{p_i\}$ *of polynomials for which* $p_i^{[k]} \to f_k$ *uniformly on* C *as* $i \to \infty$, *for* $0 \leqq k \leqq n$.

The proof of Theorem 1 will utilize the Riesz representation theorem (see Banach [1, p. 60]). If X is the set of all $n+1$ tuples $(f_0, \cdots, f_n)$ of continuous functions on C, topologized by the norm $\|(f_0, \cdots, f_n)\| = \sup \{ |f_i(z)| \mid z \in C, 0 \leqq i \leqq n\}$, then by the Riesz theorem we see that to any bounded linear functional L on X correspond unique measures $\mu_0, \cdots, \mu_n$ on $[0, 1]$ such that $L(f_0, \cdots, f_n) = \int_0^1 f_0(\phi(t)) d\mu_0(t) + \cdots + \int_0^1 f_n(\phi(t)) d\mu_n(t)$. Now let Y be the subset of X consisting of all $(p, p^{[1]}, \cdots, p^{[n]})$, where p is any polynomial. Then Theorem 1 states that Y is dense in X. By the Hahn-Banach theorem (see Banach [1]), this is equivalent to saying that every bounded linear functional on X which vanishes on Y vanishes on X. If L is the bounded linear functional in question, then by the above representation of L we see that $L(p, \cdots, p^{[n]}) = \int_0^1 p(\phi(t)) d\mu_0(t) + \cdots + \int_0^1 p^{[n]}(\phi(t)) d\mu_n(t) = 0$ for all polynomials p. To prove Theorem 1 we must show that this implies that $\mu_0 = \mu_1 = \cdots = \mu_n = 0$. The linear functional L and therefore $\mu_0, \cdots, \mu_n$ will be fixed during the discussion. As an abbreviation we set $L_t(p) = \int_0^t p(\phi(t)) d\mu_0(t) + \cdots + \int_0^t p^{[n]}(\phi(t)) d\mu_n(t)$ for all t in $[0, 1]$ and all functions p analytic in some neighborhood of C. Then $L_1(p) = 0$.

Assume now that $\phi(t) \in S$. We proceed to obtain a new formula for $L_t(p)$. To do this, take A and δ as in Definition 1, and let ϵ be positive and less than δ. Let J be the circle $|z - \phi(t)| = \epsilon$. Then by Definition 1, for z in J either $d(z, U_1) > A\epsilon^c$ or $d(z, U_2) > A\epsilon^c$, where $U_1 = \phi[0, t]$ and $U_2 = \phi[t, 1]$. Thus we can write $J = J_1 \cup J_2$, where J_1 and J_2 are disjoint Borel sets and $d(z, U_i) > A\epsilon^c$ for z in J_i. Now if p is any polynomial, let $K = \max \{ |p(z)| \mid z \in J\}$. Then

$$p(z) = (1/2\pi i) \int_J p(\zeta) d\zeta / (\zeta - z)$$

for $|z - \phi(t)| < \epsilon$. If for $i = 1$ and 2 we define

$$f_i(z) = (1/2\pi i) \int_{J_i} p(\zeta) d\zeta / (\zeta - z),$$

then we see that f_i is analytic on the complement of the closure of J_i, that $|f_i^{[j]}(z)| \leqq Kj!\, d(z, J_i)^{-(j+1)}$, and that $p(z) = f_1(z) + f_2(z)$ for $|z - \phi(t)| < \epsilon$. Thus we see that $|f_i^{[j]}(z)| < Kj! [A\epsilon^c]^{-(j+1)}$ for z in U_i.

Now let 0_1 be a neighborhood of $\phi[0, t]$ and 0_2 a neighborhood of $\phi[t, 1]$ such that $0_1 \cap 0_2 = \{z \mid |z - \phi(t)| < \epsilon\}$ and $0_1 \cap J_1 = 0_2 \cap J_2 = \phi$. Then $f_1 + f_2 = p$ on $0_1 \cap 0_2$ and f_i is analytic in 0_i. Therefore we may define an analytic function g_1 on $0_1 \cup 0_2$ by specifying $g_1(z) = f_1(z)$ for z in 0_1 and $g_1(z) = p(z) - f_2(z)$ for z in 0_2. Also define g_2 on $0_1 \cup 0_2$ by $g_2(z) = f_2(z)$ for z in 0_2 and $g_2(z) = p(z) - f_1(z)$ for z in 0_1. Then $g_1 + g_2 = p$ in $0_1 \cup 0_2$. Also $|g_i^{[j]}(z)| = |f_i^{[j]}(z)| \leqq Kj![A\epsilon^c]^{-(j+1)}$ for z in U_i, where $i = 1$ or 2 and j is arbitrary. For $i = 1$ this inequality, in conjunction with the definition of $L_t(g_1)$, tells us that $|L_t(g_1)| \leqq M_1 K \epsilon^{-c(n+1)}$, where M_1 is a constant. For $i = 2$ the inequality tells us that $|L(g_2) - L_t(g_2)| \leqq M_2 K \epsilon^{-c(n+1)}$. Thus we see that $|L_t(p)| = |L_t(g_1) + L_t(g_2)| \leqq |L_t(g_1)| + |L_t(g_2)| = |L_t(g_1)| + |L_t(g_2) - L(g_2)| \leqq MK\epsilon^{-m}$, where $M = M_1 + M_2$ and m is any integer larger than $c(n+1)$. Taking p to be the polynomial $(z - \phi(t))^j$, we see that $K = \epsilon^j$, so that $L_t([z - \phi(t)]^j) < M\epsilon^{j-m}$ for all $\epsilon < \delta$. If $j > m$, this implies that $L_t([z - \phi(t)]^j) = 0$. Therefore $L_t(p)$ depends only on the first $m+1$ terms of the expansion of p in powers on $z - \phi(t)$, so that

$$L_t(p) = \sum_{i=0}^{m} \beta_i(t) p^{[i]}(\phi(t)), \quad \text{where} \quad \beta_0(t), \cdots, \beta_m(t)$$

are certain complex numbers.

For the remainder of the proof, the only use which will be made of the fact that C satisfies a Lipschitz condition of order c at points of S will be to conclude that the expression just obtained for $L_t(p)$ is valid whenever $\phi(t)$ is in S. Therefore, the conjecture of the introductory paragraphs can be proved whenever the expression just obtained for $L_t(p)$ can be shown to be valid for a set of values of t which is dense in $[0, 1]$.

We now obtain another formula for $L_t(p)$, where now t may be any point in $(0, 1)$. For any polynomial p and any complex number z we have the Taylor's formula

$$p(z) = \sum_{i=0}^{\infty} p^{[i]}(\phi(t)) \frac{[z - \phi(t)]^i}{i!}.$$

Thus

$$\begin{aligned} L_t(p) &= \sum_{j=0}^{n} \int_0^t \left\{ \sum_{i=0}^{\infty} p^{[i+j]}(\phi(t)) \frac{[\phi(x) - \phi(t)]^i}{i!} \right\} d\mu_j(x) \\ &= \sum_{i=0}^{\infty} p^{[i]}(\phi(t)) \left(\sum_{j=0}^{i} \int_0^t \frac{[\phi(x) - \phi(t)]^{i-j}}{(i-j)!} d\mu_j(x) \right), \end{aligned}$$

where $\mu_j = 0$ if $j > n$. If we define

$$\alpha_i(t) = \sum_{j=0}^{i} \int_0^t \frac{[\phi(x) - \phi(t)]^{i-j}}{(i-j)!} d\mu_j(x),$$

for t in $(0, 1)$, we therefore have $L_t(p) = \sum_{i=0}^{\infty} p^{[i]}(\phi(t))\alpha_i(t)$. We see that α_i is continuous on the right. Comparing the two formulas obtained for $L_t(p)$, we see that $\sum_{i=0}^{m}\beta_i(t)p^{[i]}(\phi(t)) = \sum_{i=0}^{\infty}\alpha_i(t)p^{[i]}(\phi(t))$ for all t in $(0, 1)$ with $\phi(t)\in S$, and for all polynomials p. It follows that $\alpha_i(t)=0$ for $i>m$, $t\in(0, 1)$, and $\phi(t)\in S$. Since S is dense and since α_i is continuous on the right, it follows that $\alpha_i=0$ for $i>m$. Before proceeding, we need a definition.

DEFINITION 2. If f and g are two complex valued functions on $[0, 1]$, if $t\in[0, 1)$, and if a is a complex number, then a is said to be a right conditional derivative at the point t of f with respect to g if there exists a sequence $\{t_i\}$, with $t_i\in(t, 1)$, $t_i\to t$ as $i\to\infty$, and

$$[f(t_i) - f(t)][g(t_i) - g(t)]^{-1} \to a \text{ as } i \to \infty.$$

To get more information about α_i, take $0<t_0<t<1$, set $\lambda_i(t) =\mu_i[0, t]=\int_0^t d\mu_i(t)$, and consider the difference quotient

$$\{[\alpha_i(t) - \lambda_i(t)] - [\alpha_i(t_0) - \lambda_i(t_0)]\}[\phi(t) - \phi(t_0)]^{-1}$$

$$= \left\{\sum_{j=0}^{i-1}\int_0^t \frac{[\phi(x) - \phi(t)]^{i-j}}{(i-j)!} d\mu_j(x) - \sum_{j=0}^{i-1}\int_0^{t_0} \frac{[\phi(x) - \phi(t_0)]^{i-j}}{(i-j)!} d\mu_j(x)\right\}[\phi(t) - \phi(t_0)]^{-1}$$

$$= \sum_{j=0}^{i-1}\int_0^{t_0} \frac{[\phi(x) - \phi(t)]^{i-j} - [\phi(x) - \phi(t_0)]^{i-j}}{(i-j)!}[\phi(t) - \phi(t_0)]^{-1}d\mu_j(x)$$

$$+ \sum_{j=0}^{i-1}\int_{t_0<x\leqq t} \frac{[\phi(x) - \phi(t)]^{i-j}}{(i-j)!}[\phi(t) - \phi(t_0)]^{-1}d\mu_j(x).$$

The second of these summations less than

$$\sum_{j=0}^{i-1}\int_{t_0<x\leqq t} \frac{2\cdot(2B)^{i-j-1}}{(i-j)!}\,|\,d\mu_j(x)\,|$$

in absolute value, if B is the bound of $|\phi(x)|$ and if t is chosen so that $|\phi(t)-\phi(t_0)| \geqq 2^{-1}|\phi(x)-\phi(t)|$ for all x in $[t_0, t]$. The latter condition will be satisfied if $|\phi(t)-\phi(t_0)| \geqq |\phi(x)-\phi(t_0)|$ for all x in $[t_0, t]$, and values of t can be found arbitrarily close to t_0 for which this will be true. Thus the second summation can be made arbitrarily small for certain values of t close to t_0. Due to uniform convergence under the integral signs, the other of the above summations approaches

$$\sum_{j=0}^{i-1} - \int_0^{t_0} \frac{[\phi(x) - \phi(t_0)]^{i-j-1}}{(i-j-1)!} d\mu_j(x) = -\alpha_{i-1}(t_0) \text{ as } t \to t_0.$$

Thus we see that $\alpha_{j-1}(t_0)$ is a right conditional derivative at t_0 of $\lambda_j-\alpha_j$ with respect to ϕ, for $j\geqq 1$ and all t_0 in $(0, 1)$.

Since $\alpha_j=0$ for $j>m$ and since $\lambda_j=0$ for $j>n$, if $m>n$ we see that $\alpha_m(t)$, which is a right conditional derivative at t of $\lambda_{m+1}-\alpha_{m+1}=0$ with respect to ϕ, must vanish for t in $(0, 1)$. Thus $\alpha_m=0$. The argument can then be continued to show step by step that $\alpha_i=0$ for $i\geqq n$.

Therefore $\alpha_{n-1}(t)$ is a right conditional derivative at the point t of $\lambda_n-\alpha_n=\lambda_n$ with respect to ϕ, for all t in $(0, 1)$. If $\alpha_{n-1}(t)\neq 0$, this implies that $[\alpha_{n-1}(t)]^{-1}$ is a right conditional derivative at t of ϕ with respect to λ_n. Since α_{n-1} is continuous on the right, we can find $u>t$ and $r>0$ such that $|\alpha_{n-1}(x)|>r$ for x in $[t, u)$. Therefore for x in $[t, u)$ we see that $|\alpha_{n-1}(x)|^{-1}<r^{-1}$ and $[\alpha_{n-1}(x)]^{-1}$ is a right conditional derivative at x of ϕ with respect to λ_n. Hence there exist points x' arbitrarily close to x on the right with $|[\phi(x')-\phi(x)]\cdot[\lambda_n(x')-\lambda_n(x)]^{-1}|<r^{-1}$. Given any x and y in $[t, u)$, $x<y$, let T be the set of all x' in $[x, y]$ for which there exists x'' in $[x', y]$ with $|\phi(x'')-\phi(x)|\leqq r^{-1}\int_x^{x''}|d\lambda_n|$. Obviously $x\in T$. Also T is a closed subset of $[x, y]$ because $|\phi(x'')-\phi(x)|$ is a continuous function of x''. To show that T is open in $[x, y]$, take any x' in T, and choose x'' as above. If either $x'=y$ or $x'<x''$, then $[x, x'']\subset T$ is a neighborhood of x' in $[x, y]$. On the other hand, if $x'=x''<y$, then the above considerations show that there exists w in $(x', y]$ with

$$|\phi(w)-\phi(x')| < r^{-1}|\lambda_n(w)-\lambda_n(x')|.$$

Thus we have

$$\begin{aligned}|\phi(w)-\phi(x)| &\leqq |\phi(w)-\phi(x')| + |\phi(x')-\phi(x)|\\ &< r^{-1}|\lambda_n(w)-\lambda_n(x')| + r^{-1}\int_x^{x'}|d\lambda_n| \leqq r^{-1}\int_x^{w}|d\lambda_n|.\end{aligned}$$

Therefore $w\in T$ so that $[x, w]\subset T$. Thus $[x, w]$ is a neighborhood of x' in $[x, y]$. Hence T is both open and closed in $[x, y]$. Since $x\in T$, $T=[x, y]$. Therefore $y\in T$, so that $|\phi(y)-\phi(x)|\leqq r^{-1}\int_x^y|d\lambda_n|$ for all x and y in $[t, u)$. Therefore ϕ has bounded variation on $[t, u]$, so that $\phi[t, u]$ is a rectifiable sub-arc of C. This contradicts the hypothesis. This contradiction shows that $\alpha_{n-1}(t)=0$ for all t in $(0, 1)$, so that $\alpha_{n-1}=0$. Having proved this, we can use the same argument to show step by step that $\alpha_{n-2}=\alpha_{n-3}=\cdots=\alpha_0=0$. But $\alpha_0(t)=\int_0^t d\mu_0(t)$. Thus μ_0 vanishes on all subsets of $[0, 1)$. Since there is inherent symmetry between the endpoints, $\mu_0=0$. Then

$$0=\alpha_1(t)=\int_0^t[\phi(x)-\phi(t)]d\mu_0(x)+\int_0^t d\mu_1(x)=\int_0^t d\mu_1(x),$$

so that $\mu_1=0$. Thus we show step-by-step that $\mu_0=\mu_1=\cdots=\mu_n=0$. This completes the proof of Theorem 1.

There exists Jordan arcs for which condition (2) is fulfilled. For instance, a Jordan arc which has no rectifiable sub-arcs and which has a tangent at a dense set of points will do, because the existence of a tangent implies that a Lipschitz condition of order 1 is fulfilled. To see this, assume that C has a tangent at $\phi(t_0)$. By this we mean that the parameter t can be so chosen that $\phi'(t_0)$ exists and is not zero. Now if C does not satisfy a Lipschitz condition of order 1 at $\phi(t_0)$, then for each $\delta>0$ there exists z with $|\phi(t_0)-z|<\delta$ such that $d(\phi[0, t_0], z)<|\phi(t_0)-z|/4$ and $d(\phi[t_0, 1], z)<|\phi(t_0)-z|/4$. Therefore, there exist t_1 in $[0, t_0]$ and t_2 in $[t_0, 1]$ with

$$|\phi(t_1) - z| < |\phi(t_0) - z|/4$$

and $|\phi(t_2)-z|<|\phi(t_0)-z|/4$, so that $|\phi(t_1)-\phi(t_2)|<|\phi(t_0)-z|/2$. Also,

$$\begin{aligned} &|\phi(t_0) - z| \\ &\leqq |\phi(t_0) - \phi(t_1)| + |(t_1) - z| < |\phi(t_0) - \phi(t_1)| + |\phi(t_0) - z|/4, \end{aligned}$$

so that $3|\phi(t_0)-z|/4<|\phi(t_0)-\phi(t_1)|$. Thus,

$$|\phi(t_1) - \phi(t_2)| < 2|\phi(t_0) - \phi(t_1)|/3.$$

It follows that

$$\gamma = |\phi(t_1) - \phi(t_2)|\,|t_1 - t_2|^{-1}[|\phi(t_0) - \phi(t_1)|\,|t_0 - t_1|^{-1}]^{-1} < 2/3.$$

On the other hand, as $\delta\to 0$ the quantity γ converges to $|\phi'(t_0)| \cdot |\phi'(t_0)|^{-1}=1$. This contradiction shows that C satisfies a Lipschitz condition of order 1 at $\phi(t_0)$. To construct a Jordan arc which has no rectifiable sub-arcs and which has a tangent at a dense set of points, let f be any continuous real function on $[0, 1]$ such that f' exists at a dense set S of points and such that in any sub-interval the set of points where f' does not exist has positive measure. Let $\phi(t)=t+if(t)$, so that $\phi[0, 1]=C$ is a Jordan arc having a tangent at the dense set of points $\phi(S)$. Also C has no rectifiable sub-arcs, because if $\phi[t, u]$ were a rectifiable sub-arc then ϕ would be of bounded variation on $[t, u]$, which would imply that f' would exist almost everywhere in $[t, u]$, contrary to the condition on f.

It only remains to construct the function f. The standard techniques for the construction of nondifferentiable functions can be used (see Hobson [**2**]). Let $\{r_n\}=s$ be a sequence of irrational numbers which is dense in $[0, 1]$. It is easy to construct inductively a sequence $\{C_n\}$ of countable subsets of $[0, 1]$, each consisting of rational numbers and each containing 0 and 1, such that the accumulation points of C_n are exactly $r_1, \cdots, r_n$, such that the distance of two consecutive points t_1 and t_2 of C_n is not larger than $6^{-n}d$, where d is the distance between the sets $\{t_1, t_2\}$ and $\{r_1, \cdots, r_n\}$, and such that $D_n \subset C_{n+1}$. Here $D_n = C_n \cup \tilde{C}_n$, where $\tilde{C}_n$ consists of all points which lie midway between two consecutive points of C_n. Thus D_n consists of rational numbers, and therefore is disjoint from the sequence s. Define the function f_n on $[0, 1]$ as follows. Let $f_n(t) = 0$ if $t \in C_n \cup \{r_1, \cdots, r_n\}$. For consecutive points t_1 and t_2 in C_n, let $f_n(t) = 3^n(t - t_1)$ if $t_1 \leqq t \leqq (t_1 + t_2)/2$ and $f_n(t) = 3^n(t_2 - t)$ if $(t_1 + t_2)/2 \leqq t \leqq t_2$. Since any point of $[0, 1]$ which is not in $C_n \cup \{r_1, \cdots, r_n\}$ lies between two consecutive points of C_n, this defines f_n uniquely at all points of $[0, 1]$. The function f_n is clearly continuous, except possibly at the points $r_1, \cdots, r_n$, and $|f_n'(t)|$ exists and equals 3^n if t is not in the set $D_n \cup \{r_1, \cdots, r_n\}$. Also, if u_1 and u_2 are consecutive points of D_n, it is clear that $|f_n(u_2) - f_n(u_1)|(u_2 - u_1)^{-1} = 3^n$. To investigate the behavior of f_n at the point r_i, where $i \leqq n$, let t be any point of $[0, 1] - \{r_1, \cdots, r_n\}$. There exist consecutive points t_1 and t_2 of C_n with $t_1 \leqq t \leqq t_2$. By the definition of f_n, we have $|f_n(t)| \leqq 3^n(t_2 - t_1)$. By the construction of C_n, we have $t_2 - t_1 \leqq 6^{-n}|t - r_i|^2$. Thus, $|f_n(t) - f_n(r_i)| = |f_n(t)| \leqq 2^{-n}|t - r_i|^2$ for all t in $[0, 1]$. In particular, we see that $f_n'(r_i) = 0$, so that $f_n'(t)$ exists if t is not in D_n, and that f_n is continuous at r_i. Therefore, f_n is continuous on $[0, 1]$. We also see that $0 \leqq f_n(t) \leqq 2^{-n}$ for all t. The series $\sum_{n=1}^{\infty} f_n$ therefore converges uniformly on $[0, 1]$ to a continuous function f. It will be shown that $f'(t)$ exists if and only if t is a member of the sequence s. If $t = r_i$ is a member of s, let $f = g + h$, where $g = \sum_{n=1}^{i-1} f_n$ and $h = \sum_{n=i}^{\infty} f_n$. Then $g'(r_i)$ exists because r_i is not in D_n for all n. Also,

$$|h(u) - h(r_i)| \leqq \sum_{n=i}^{\infty} |f_n(u) - f_n(r_i)| \leqq \sum_{n=i}^{\infty} 2^{-n}|u - r_i|^2 \leqq |u - r_i|^2$$

for all u, so that $h'(r_i)$ exists and is zero. Therefore $f'(r_i)$ exists. Now assume that t is not an element of the sequence s. Let n be arbitrary. Since t is not an accumulation point of C_n, there exist consecutive points u_1 and u_2 in D_n with $u_1 \leqq t \leqq u_2$. Then $f(u_2) - f(u_1) = \sum_{k=1}^{n} f_k(u_2) - f_k(u_1)$ because $f_k(u) = 0$ if $u \in D_n \subset C_{n+1}$ and $k \geqq n+1$. Thus

$$\begin{aligned} &|f(u_2) - f(u_1)|(u_2 - u_1)^{-1} \\ &\quad \geqq |f_n(u_2) - f_n(u_1)|(u_2 - u_1)^{-1} - \sum_{k-1}^{n-1} |f_k(u_2) - f_k(u_1)|(u_2 - u_1)^{-1} \\ &\quad \geqq 3^n - \sum_{k=1}^{n} 3^k \geqq n. \end{aligned}$$

Thus the difference quotients, $[f(u_2) - f(u_1)](u_2 - u_1)^{-1}$, for $u_1 \leqq t \leqq u_2$, are not bounded. It follows that $f'(t)$ does not exist, as was to be proved.

References

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AN IDENTITY IN THE THEORY OF THE GENERALIZED POLYNOMIALS OF JACOBI

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1. Introduction of some new notations in the theory of the Jacobi polynomials. To facilitate the passage from the usual Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ to the generalized Jacobi polynomials $P_n^{(\alpha_0,\cdots,\alpha_p)}(x)$ considered here, we introduce some new notations in the theory of the first mentioned polynomials. It is well known[1] that the zeros of these polynomials are the points $x_1=x_1^{(n)}$, $x_2=x_2^{(n)}$, $\cdots$, $x_n=x_n^{(n)}$, which maximize the expression

$$T(x_1, x_2, \cdots, x_n) = T(x) = \prod_{k=1}^{n} (1-x_k)^p(1+x_k)^q \prod_{1\leq\nu<\mu\leq n} |x_\nu - x_\mu|$$

in the unit-interval I: $[-1, +1]$. Here $\alpha=2p-1$ and $\beta=2q-1$ and it is assumed $x_1>x_2> \cdots >x_n$. Instead of $T(x)$ we use the expression

$$V_m(\xi_1, \xi_2, \cdots, \xi_m; e_1, e_2, \cdots, e_m) = V_m(\xi; e) = \prod_{1\leq i<k\leq m} (\xi_i - \xi_k)^{e_ie_k},$$

where we suppose that $m=n+2$; that the points ξ_1 and ξ_m are fixed from the outset and are equal to $a_0=-1$ and $a_1=+1$ respectively; that $e_1=\rho_0=q$, $e_m=\rho_1=p$, $e_2=e_3= \cdots =e_{m-1}=1$; that the points $\xi_1, \xi_2, \cdots, \xi_m$ are counted in increasing order; $-1=\xi_1<\xi_2< \cdots <\xi_{m-1}<\xi_m=+1$ and therefore that $\xi_2=x_n$, $\xi_3=x_{n-1}$, $\cdots$, $\xi_{m-1}=x_1$. It results that $V_m(\xi_1, \xi_2, \cdots, \xi_m; e_1, e_2, \cdots, e_m)$ is a function of $\xi_2, \xi_3, \cdots, \xi_{m-1}; \rho_0, \rho_1$ only, as is $T(x)$. Then the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ are the points $\xi_2=\xi_2^{(m)}=x_n^{(n)}$, $\xi_3=\xi_3^{(m)}=x_{n-1}^{(n)}$, $\cdots$, $\xi_{m-1}=\xi_{m-1}^{(m)}=x_1^{(n)}$, which maximize the absolute value of $V_m(\xi; e)$ on I, under the mentioned conditions. We call the last function the *generalized Vandermondean* of the degree m and of the order 1. We write

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[1] Szegö, *Orthogonal polynomials*, New York, 1939, p. 136, Theorem 6.7.1.