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UNIVERSITY OF WASHINGTON

NOTE ON PRODUCTS IN Ext

NOBUO YONEDA[1]

The objective of this note is to present an interrelation between the $\vee$-product in [1] and the composition product in [2; 3], which in turn gives a comparison of the cup-product with the composition product. Similar relations can be obtained also for other products involving Tor and (iterated) connecting homomorphisms.

We retain the notations in [1, Chapter XI]. The external product

$$\vee\colon \operatorname{Ext}_{\Lambda\otimes\Sigma^*}(A, C) \otimes \operatorname{Ext}_{\Sigma\otimes\Gamma^*}(A', C') \to \operatorname{Ext}_{\Lambda\otimes\Gamma^*}(A \otimes_\Sigma A', C \otimes_\Sigma C')$$

is defined in the situation $({}_\Lambda A_\Sigma, {}_\Lambda C_\Sigma, {}_\Sigma A'_\Gamma, {}_\Sigma C'_\Gamma)$ under the following assumption: (i) Λ, Γ, Σ are K-projective; (ii) $\operatorname{Tor}_n^\Sigma(A, A')=0$ for $n>0$. To this situation we now add $({}_\Lambda B_\Sigma, {}_\Sigma B'_\Gamma)$ and (ii′) $\operatorname{Tor}_n^\Sigma(B, B')=0$ for $n>0$. For $a\in\operatorname{Ext}^p_{\Lambda\otimes\Sigma^*}(A, B)$ and $b\in\operatorname{Ext}^q_{\Lambda\otimes\Sigma^*}(B, C)$ the composition product $b \circ a$ lies in $\operatorname{Ext}^{p+q}_{\Lambda\otimes\Sigma^*}(A, C)$. For $a'\in\operatorname{Ext}^{p'}_{\Sigma\otimes\Gamma^*}(A', B')$ and $b'\in\operatorname{Ext}^{q'}_{\Sigma\otimes\Gamma^*}(B', C')$, $b' \circ a'$ lies in $\operatorname{Ext}^{p'+q'}_{\Sigma\otimes\Gamma^*}(A', C')$. ($b \circ a = a \bigcirc b$ in the notation of [2].)

PROPOSITION 1. $(b \circ a)\vee(b' \circ a') = (-1)^{pq'}(b\vee b') \circ (a\vee a')$.

In fact let X, Y be $\Lambda\otimes\Sigma^*$-projective resolutions of A, B respectively, and let X', Y' be $\Sigma\otimes\Gamma^*$-projective resolutions of A', B'. Then $X\otimes_\Sigma X'$, $Y\otimes_\Sigma Y'$ are $\Lambda\otimes\Gamma^*$-projective resolutions of $A\otimes_\Sigma A'$, $B\otimes_\Sigma B'$. We consider these resolutions as chain complexes with 0's in negative dimensions. Suppose that a, b, a', b' are respectively represented by maps $\alpha\colon X_p\to B$, $\beta\colon Y_q\to C$, $\alpha'\colon X'_{p'}\to B'$, and $\beta'\colon Y'_{q'}\to C'$. The map α

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is lifted to a chain map $\tilde{\alpha}_n: X_{p+n} \to Y_n$ of degree $-p$, and α' to $\tilde{\alpha}_n': X'_{p'+n} \to Y_n'$ of degree $-p'$. The elements $b \circ a$, $b' \circ a'$ are then represented by $\beta\tilde{\alpha}_q: X_{p+q} \to C$ and $\beta'\tilde{\alpha}'_{q'}: X'_{p'+q'} \to C'$, so that $(b \circ a) \vee (b' \circ a')$ is represented by the map $\beta\tilde{\alpha}_q \otimes_\Sigma \beta'\tilde{\alpha}'_{q'}: (X \otimes_\Sigma X')_{p+q+p'+q'} \to C \otimes_\Sigma C'$. On the other hand $a \vee a'$, $b \vee b'$ are represented by the maps

$$\alpha \otimes_\Sigma \alpha': (X \otimes_\Sigma X')_{p+p'} \to B \otimes_\Sigma B', \quad \beta \otimes_\Sigma \beta': (Y \otimes_\Sigma Y')_{q+q'} \to C \otimes_\Sigma C'.$$

We now define maps $\tilde{\alpha}_n'': (X \otimes_\Sigma X')_{p+p'+n} \to (Y \otimes_\Sigma Y')_n$ by

$$\tilde{\alpha}_n'' = \sum_i (-1)^{p(n-i)} \tilde{\alpha}_i \otimes_\Sigma \tilde{\alpha}'_{n-i}.$$

Then $\tilde{\alpha}''$ is a chain map $(X \otimes_\Sigma X') \to (Y \otimes_\Sigma Y')$ of degree $-(p+p')$, and it gives a lift of the map $\alpha \otimes_\Sigma \alpha'$. Therefore $(b \vee b') \circ (a \vee a')$ is represented by the map $(\beta \otimes_\Sigma \beta')\tilde{\alpha}''_{q+q'} = (-1)^{pq'} \beta\tilde{\alpha}_q \otimes_\Sigma \beta'\tilde{\alpha}'_{q'}$, which proves our assertion.

Instead of (i), (ii) we shall now assume (iii-A) A is Σ-flat. Let X' be as before, and let X'' be a $\Lambda \otimes \Gamma^*$-projective resolution of $A \otimes_\Sigma A'$. Since $A \otimes_\Sigma X'$ is acyclic over $A \otimes_\Sigma A'$, we have a chain map $X'' \to A \otimes_\Sigma X'$ lifting the identity map of $A \otimes_\Sigma A'$, which induces a cochain map

$$\mathrm{Hom}_{\Lambda \otimes \Gamma^*}(A \otimes_\Sigma X', A \otimes_\Sigma C') \to \mathrm{Hom}_{\Lambda \otimes \Gamma^*}(X'', A \otimes_\Sigma C').$$

On the other hand $A \otimes_\Sigma$ induces naturally a cochain map

$$\mathrm{Hom}_{\Sigma \otimes \Gamma^*}(X', C') \to \mathrm{Hom}_{\Lambda \otimes \Gamma^*}(A \otimes_\Sigma X', A \otimes_\Sigma C').$$

By composing these cochain maps we get a homomorphism

$$A \otimes_\Sigma: \mathrm{Ext}_{\Sigma \otimes \Gamma^*}(A', C') \to \mathrm{Ext}_{\Lambda \otimes \Gamma^*}(A \otimes_\Sigma A', A \otimes_\Sigma C').$$

If an element $c' \in \mathrm{Ext}^n_{\Sigma \otimes \Gamma^*}(A', C')$ is represented by an n-fold extension ϕ^n of C' over A' (cf. [**2**; **3**]), then $A \otimes_\Sigma c'$ is represented by the exact sequence $A \otimes_\Sigma \phi^n$. A similar homomorphism

$$\otimes_\Sigma C': \mathrm{Ext}_{\Lambda \otimes \Sigma^*}(A, C) \to \mathrm{Ext}_{\Lambda \otimes \Gamma^*}(A \otimes_\Sigma C', C \otimes_\Sigma C')$$

is obtained when (iii-C') C' is Σ-flat. These homomorphisms preserve the composition product. By a similar technique as in the proof of Proposition 1, we get easily the following:

PROPOSITION 2. *Under the assumptions* (i) *and* (iii-A) *we have* $A \otimes_\Sigma = e_A \vee$, *where* e_A *is the identity map of* A *considered as an element of* $\mathrm{Ext}^0_{\Lambda \otimes \Sigma^*}(A, A)$. *Under the assumptions* (i) *and* (iii-C') *we have* $\otimes_\Sigma C' = \vee e_{C'}$.

We now define products $\vee_0$, $\vee^0$ parallel to $\vee$ by

$$c \vee_0 c' = (c \otimes_\Sigma C') \circ (A \otimes_\Sigma c'), \quad c \vee^0 c' = (C \otimes_\Sigma c') \circ (c \otimes_\Sigma A').$$

The former is defined under the assumptions (iii-A), (iii-C'), and the latter under (iii-C), (iii-A'). From Propositions 1, 2 we obtain:

PROPOSITION 3. *For* $c \in \mathrm{Ext}^n_{\Lambda\otimes\Sigma^*}(A, C)$, $c' \in \mathrm{Ext}^{n'}_{\Sigma\otimes\Gamma^*}(A', C')$ *we have*

$$c \vee_0 c' = c \vee c' \text{ when both sides are defined;}$$

$$c \vee^0 c' = (-1)^{nn'} c \vee c' \text{ when both sides are defined.}$$

Finally we consider the cup-product, and restrict our attention to the case where Λ is a supplemented K-algebra with augmentation $\epsilon: \Lambda \to K$, and where $\Gamma = \Lambda^*$, $\Sigma = K$. For a "diagonal map" $D: \Lambda \to \Lambda \otimes \Lambda$ we postulate the relation $(\epsilon \otimes \Lambda)D = (\Lambda \otimes \epsilon)D =$ identity map of Λ. We shall denote by $D\#$ the natural map

$$D\#: \mathrm{Ext}_{\Lambda\otimes\Lambda}(A \otimes A', C \otimes C') \to \mathrm{Ext}_\Lambda (A \otimes A', C \otimes C')$$

induced by D.

We say "Λ operates trivially on A" if Λ operates on A only through ϵ. $A \otimes A'$ as a left Λ-module does not depend on the diagonal map D if Λ operates trivially on A or on A'. Thus if Λ operates trivially on A and on C', or on C and on A', then $D\#(c \vee_0 c')$ or $D\#(c \vee^0 c')$ does not depend on D. So we get:

PROPOSITION 4. *If A, C' are K-flat and Λ operates trivially on those, or if C, A' are K-flat and Λ operates trivially on those, then the cup-product $c \cup c'$ does not depend on D.*

Noting the fact that $D\#(c \otimes K) = c$, $D\#(K \otimes c') = c'$ we get:

PROPOSITION 5. *Let Λ be a K-projective supplemented K-algebra. Then the composition product*

$$\circ: \mathrm{Ext}_\Lambda (K, K) \otimes \mathrm{Ext}_\Lambda (K, K) \to \mathrm{Ext}_\Lambda (K, K)$$

concides with the cup-product induced by any diagonal map of Λ. It is associative and anti-commutative.

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UNIVERSITY OF TOKYO