A CLASS OF UNKNOTTED CURVES IN 3-SPACE

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1. **Introduction.** If S is a 2-sphere regularly imbedded (see below for definition) in 3-space E^3, the union of S and the bounded component of the complement of S is a 3-cell (this component will be called the interior of S). If C is the homeomorphic image of a circle lying on S, it is clear there is a topological disc D whose interior points lie in the interior of S and whose boundary is C. Similarly if $C_1, C_2, \cdots, C_n$ are pairwise disjoint homeomorphic images of a circle all lying on S, then there are pairwise disjoint discs $D_1, D_2, \cdots, D_n$ with boundaries $C_1, C_2, \cdots, C_n$ and whose interior points are interior to S.

In this paper a generalization of this result is proved. We define a certain class of imbeddings of a closed connected 2-dimensional surface M in E^3. For this class of imbeddings if $C_1, \cdots, C_n$ are pairwise disjoint homeomorphs of a circle, each one separating M into two components, then there are pairwise disjoint discs $D_1, \cdots, D_n$ with boundaries $C_1, \cdots, C_n$ and whose interior points are interior to M.

2. **Definitions.** By E^3 we mean euclidean 3-space together with a fixed coordinate system (x, y, z). E^2 will be the subset of E^3 consisting of the points with $z=0$. E^2 will have the induced (x, y) coordinate system. S will be the unit 2-sphere in E^3, that is the subset defined by the equation $x^2+y^2+z^2=1$. We assume S has the usual differential structure and an orientation induced by that on E^3.

C and D will be the subsets of E^2 defined respectively by the equations $x^2+y^2=1$ and $x^2+y^2\leqq 1$. D is assumed orientated as E^2 and C has the orientation coherent with that of D. In certain connections we will use polar coordinates $re^{i\theta}$ for (x, y) when dealing with points of D.

T will denote the torus in E^3 defined by the parametric equations $x=(2+\cos\phi)\cos\theta, y=(2+\cos\phi)\sin\theta, z=\sin\phi$ $(0\leqq\theta\leqq 2\pi; 0\leqq\phi\leqq 2\pi)$. T will be assumed to have the usual differential structure and an orientation coherent with that of E^3. T^* will be the subset of T consisting of those points (x, y, z) with $x\leqq 3/2$. It is given the differential structure and orientation of T. $\dot{T}^*$ is the topological boundary of T^*. It is the intersection of T^* with the plane $x=3/2$ and is homeomorphic with C.

If B is one of the above defined spaces, by a regular mapping of

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B into E^3 we mean a differentiable homeomorphism $f: B \to E^3$ such that the jacobian has maximal rank at each point. Unless otherwise noted, all mappings considered will be regular in the above sense. The term "mapping" will be used for "regular mapping" when there is little danger of misunderstanding.

It will be convenient to identify a point p of E^3 with the vector from the origin to p. With this convention the points kp (k a real number) and $p+q$ are defined in the obvious fashion. The length of a vector v will be denoted by $|v|$. $|p|$ will be the distance from p to the origin.

DEFINITION 1. By a circle in E^3 we mean a mapping $f: C \to E^3$. The image set $f(C)$ will also be called a circle.

DEFINITION 2. By a disc in E^3 we mean a mapping $g: D \to E^3$. The image set $g(D)$ will also be called a disc.

DEFINITION 3. A circle $f(C)$ is said to be the boundary of a disc $g(D)$ if $g|D = f: C$.

DEFINITION 4. Let I be the interval $[0, 1]$. By an interval we mean any mapping $f: I \to E^3$. The point set $f(I)$ will also be called an interval.

We are now in a position to define the class of surfaces in E^3 which is studied in our theorem.

DEFINITION 5. Let $e: S \to E^3$ be a mapping and let $\tilde{g}_i: D \to e(S)$ $(i=1, \cdots, n)$ be mappings such that $\tilde{g}_i(D) \cap \tilde{g}_j(D) = \varnothing$ $(i \neq j)$. Let $h_i: T^* \to E^3$ $(i=1, \cdots, n)$ be mappings such that $h_i(T^*) \cap h_j(T^*) = \phi (i \neq j)$, $h_i(T^* - \dot{T}^*)$ is in the unbounded component of the complement of $e(S)$ and $h_i(\dot{T}^*) = \tilde{g}_i(C)$. The normal vectors to $e(S)$ and $h_i(T^*)$ will be assumed continuous across $g_i(C)$. The set

$$e(S) \cup \bigcup_{i=1}^{n} h_i(T^*) - \bigcup_{i=1}^{n} \tilde{g}_i(D - C)$$

will be called a surface of class A.

In the remainder of the paper M_n will denote a surface of class A with n "handles," that is a sphere with n holes and images of T^* attached to these holes as in Definition 5. The bounded component of the complement of M_n will be called the interior of M_n.

The point set $h_i(T^*) \cup g_i(D)$ is a topological torus for each i. It is not smoothly imbedded in E^3, however, because of the edge $\tilde{g}_i(C)$.

3. **The main result.** We have the following

THEOREM. *Let $\{f_i(C)\}$ $(i=1, \cdots, m)$ be a collection of circles with $f_i(C) \subset M_n$ and $f_i(C) \cap f_j(C) = \phi$ $(i \neq j)$. Suppose each $f_i(C)$ is homologous to zero in M_n. Then there are discs $g_i(D)$ $(i=1, \cdots, m)$ such that*

$g_i | C = f_i$: C and $g_i | D - C$ is interior to M_n. Further $g_i(D) \cap g_j(D) = \phi$ $(i \neq j)$.

We remark that for a circle on M_n to be homologous to zero is equivalent to separating M_n into two connected pieces.

The main argument of the proof will be preceded by further definitions and some lemmas.

4. **Proof of some lemmas.** Two circles $f^0(C)$ and $f^1(C)$ will be called isotopic in E^3 if there is a differentiable mapping F: $C \times I \rightarrow E^3$ such that $F | C \times 0 = f^0$: C and $F | C \times 1 = f^2$: C and for each $t \in I$, $F | C \times t$ is a circle. F need not be a regular mapping in the sense we have defined above. The isotopy of two discs is defined similarly. $f^0(C)$ and $f^1(C)$ are isotopic in M_n if $F(C \times I) \subset M_n$.

Let $\{f_i^0(C)\}$ $(i = 1, \cdots, m)$ be a set of pairwise disjoint circles on M_n. $\{f_i^0(C)\}$ is isotopic to the set of circles $\{f_i^1(C)\}$ if there are isotopies F_i: $C \times I \rightarrow M_n$ with $F_i | C \times 0 = f_i^0$: C, $F_i | C \times 1 = f_i^1$: C and for each t, the set of circles $\{F_i | C \times t\}$ is pairwise disjoint.

Lemma 1. *Let $\{f_i^0(C)\}$ and $\{f_i^1(C)\}$ $(i = 1, \cdots, m)$ be two pairwise disjoint sets of circles on M_n which are isotopic on M_n. Suppose there is a pairwise disjoint set of discs $\{g_i^0(D)\}$ such that $f_i^0(C)$ is the boundary of $g_i^0(D)$ $(i = 1, \cdots, m)$ and such that $\bigcup_{i=1}^m (g_i^0(D) - g_i^0(C))$ is in the interior of M_n. Then there is a pairwise disjoint set of discs $\{g_i^1(D)\}$ with boundaries $\{f_i^1(C)\}$ such that $\bigcup_{i=1}^m (g_i^1(D) - g_i^1(C))$ is in the interior of M_n.*

Proof. From Whitney [2, p. 667], there exists a vector field $N(p)$ which is a differentiable function of p and which for each p sticks into the interior of M_n ($N(p)$ is to be thought of as originating at p). It is clear the $N(p)$ can be chosen of equal length and short enough so that the equation $p + sN(p) = p' + s'N(p')$ implies $p = p'$ and $s = s'$ if $0 \leqq s$, $s' \leqq 1$. Clearly the $g_i^0(D)$ may be chosen so that $g_i^0((1-s)p) = f_i^0(p) + sN(f_i^0(p))$ $(i = 1, \cdots, m)$ for all $p \in C$ and all $0 \leqq s \leqq 2\delta$ for some δ, $0 < \delta < 1/2$. Again by a suitable choice of δ we may assume the only points of $g_i^0(D)$ within a distance $2\delta | N(p) |$ of M_n are those of the form $g_i^0((1-s)p)$ where $p \in C$ and $0 \leqq s \leqq 2\delta$.

Suppose then F_i: $C \times I \rightarrow M_n$ is an isotopy of $f_i^0(C)$ into $f_i^1(C)$. Define mappings $\bar{g}_i$: $D \rightarrow E^3$ as follows.

$$(*) \qquad \begin{aligned} \bar{g}_i(re^{i\theta}) &= g_i(re^{i\theta}) \text{ for } 0 \leqq r \leqq 1 - 2\delta. \\ \bar{g}_i(re^{i\theta}) &= F_i(e^{i\theta}, t(r)) + m(r)N(F(e^{i\theta}, t(r))) \end{aligned}$$

where $t(r) = ar^3 + br^2 + cr + d$, $m(r) = Ar^3 + Br^2 + Cr + D$ and

$$\begin{aligned} a &= -2/\delta^3, & A &= -4/\delta^2, \\ b &= (6-9\delta)/\delta^3, & B &= (12-18\delta)/\delta^2, \\ c &= (-6+18\delta-12\delta^2)/\delta^3, & C &= (-12+36\delta-25\delta^2)/\delta^2, \\ d &= (2-9\delta+12\delta^2-4\delta^3)/\delta^3, & D &= (1-18\delta+26\delta^2-12\delta^3)/\delta^2 \end{aligned}$$

for $1-2\delta \leqq r \leqq 1-\delta$,

$$\bar{g}(re^{i\theta}) = f_i^1(e^{i\theta}) + (1-r)N(f_i^1(e^{i\theta}))$$

for $1-\delta \leqq r \leqq 1$.

The equations ($*$) define a set of discs satisfying the conditions of Lemma 1. It can be checked directly that $\bar{g}_i$ is a continuously differentiable mapping whose jacobian is of maximum rank everywhere (i.e. $\bar{g}_i$ is an immersion of D into E^3). We must show that $\bar{g}_i$ is a homeomorphism, the $\{\bar{g}_i(D)\}$ are pairwise disjoint and $\bar{g}_i|D-C$ is contained in the interior of M_n.

The restriction of $\bar{g}_i$ to points $re^{i\theta}$ with $0 \leqq r \leqq 1-2\delta$ is a homeomorphism and the images are pairwise disjoint since on this subset of D, $\bar{g}_i$ coincides with g_i. The only points of $\bar{g}_i(D)$ within $2\delta|v(p)|$ of M_n (as measured along the segment $p+\lambda v(p)$ passing through the point p) are those of the form $\bar{g}_i(re^{i\theta})$ with $1-2\delta \leqq r \leqq 1$. The set of points of $\{\bar{g}_i(D)\}$ at a given distance from M_n along $v(p)$ is a translation (without intersection) of a set of curves $\{F_i|C\times t\}$ $(0 \leqq t \leqq 1)$. But these curves are circles and are pairwise disjoint. From this it is evident the $\{\bar{g}_i(D)\}$ satisfy the first two conditions above. That $\bigcup_{i=1}^m g_i(D-C)$ is contained in the interior of M_n is obvious. This proves Lemma 1.

LEMMA 2. *Let $g'(D)$ be a disc contained in the interior of a disc $g''(D)$. Let $N(p)$ be a continuously differentiable vector field which for each $p \in g''(D)$ has a nonzero projection on the normal to $g''(D)$ at p. Then there is a disc $g(D)$ with $g(D) \cap g''(D) = g'(C)$ and $g(p) = g'(p') + \lambda(p)N(g'(p'))$ for each $p \in D$. $\lambda(p)$ is a real function with $0 \leqq \lambda(p) \leqq 1$ and p' is a point in D depending on p. $\lambda(p)$ can be chosen so that given $\epsilon > 0$, $\max_{p\in D} \lambda(p)|N(p')| < \epsilon$.*

PROOF. We define the disc $g(D)$ by the following equations:

$$g(re^{i\theta}) = g'(t(r)e^{i\theta}) + m(r)N(g'(t(r)e^{i\theta}))$$

where $t(r)$ and $m(r)$ are the functions of r defined below.

$$t(r) = 1 \text{ for } 1-\delta \leqq r \leqq 1.$$

$$t(r) = ar^3 + br^2 + cr + d \text{ for } 1-2\delta \leqq r \leqq 1-\delta.$$

$$t(r) = [(1-\delta)/(1-2\delta)]r \text{ for } 0 \leqq r \leqq 1-2\delta.$$

$$m(r) = 1 - r \text{ for } 1 - \delta \leqq r \leqq 1.$$
$$m(r) = Ar^3 + Br^2 + Cr + D \text{ for } 1 - 2\delta \leqq r \leqq 1 - \delta.$$
$$m(r) = 2\delta \text{ for } 0 \leqq r \leqq 1 - 2\delta.$$

The coefficients for the cubic portions of $t(r)$ and $m(r)$ are

$$\begin{aligned}
a &= (3\delta - 1)/\delta^2(1 - 2\delta), & A &= 1/\delta^2,\\
b &= (14\delta^2 - 14\delta + 3)/\delta^2(1 - 2\delta), & B &= (4\delta - 3)/\delta^2,\\
c &= (19\delta^3 - 35\delta^2 + 19\delta - 3)/\delta^2(1 - 2\delta), & C &= (4\delta^2 - 8\delta - 3)/\delta^2,\\
d &= (-4\delta^3 + 10\delta^2 - 6\delta + 1)/\delta^2, & D &= (2\delta^3 - 4\delta^2 + 4\delta - 1)/\delta^2.
\end{aligned}$$

It can be verified directly that $g(D)$ is a disc satisfying the conditions of the lemma. By choosing δ sufficiently small one can insure that $\max_{p\in D} \lambda(p)|N(p')| < \epsilon$ ($\lambda(p) = m(r)$ in our formulas).

LEMMA 3. *Let $h(M)$ be a topological imbedding of a differentiable 2-manifold M in E^3. Suppose $f(C) \subset M$ and h is such that $h(M)$ is regular except along $f(C)$. On $f(C)$, $h(M)$ has a family of cusps, that is if $P(p)$ is the normal plane to $h(f(C))$ at $p \in h(f(C))$ then in a neighborhood of p in $P(p)$, $P(p) \cap h(M)$ consists of two regular intervals with just p in common. These intervals will have distinct tangents at p in general. Under these conditions given $\delta_1 > 0$ there is an imbedding $h'(M) \subset E^3$ such that $|h'(p) - h(p)| < \delta_1$, for all $p \in M$. Assuming M has a metric, given $\delta_2 > 0$ we may determine h' so that $h'(p) = h(p)$ except for $|p - f(C)| < \delta_2$. Finally, h' can be chosen so that for points p with $h'(p) \neq h(p)$, $h'(p)$ is in the interior of $h(M)$.*

Let $P(p)$ ($p \in h(f(C))$) be a continuously differentiable family of planes which for each $p \in h(f(C))$ is independent of the tangent line to $h(f(C))$ at p. In a neighborhood of p, $P(p) \cap h(M)$ consists of two intervals $L_1(p)$ and $L_2(p)$ and $h^{-1}(L_1(p) \cup L_2(p))$ is a regular interval in M which is transverse to $f(C)$. h is modified in a neighborhood of $f(C)$ by mapping $h^{-1}(L_1(p) \cup L_2(p))$ into $N(p)$ and fitting together the mappings on the various $h^{-1}(L_1(p) \cup L_2(p))$ in a continuously differentiable fashion.

We indicate how the latter can be done. The exact formulas are complicated and will not be reproduced here. A coordinate system is chosen for each $P(p)$ by choosing the x-axis along the tangent vector to $L_1(p)$ and the y-axis normal to this x-axis. Two points $p_1(p) \in L_1(p)$, $p_2(p) \in L_2(p)$ are chosen such that $p_1(p)$ and $p_2(p)$ are both distinct from p and their distances from p are differentiable functions of p. Let $T_1(p_1(p))$ and $T_2(p_2(p))$ be the tangent rays to $L_1(p)$ and $L_2(p)$ at the points p_1 and p_2 respectively which point in the direction of p. Let $T_3(p)$ be a line in $P(p)$ which cuts across T_1 and T_2 (notice that $T_1 \neq T_2$), is "near" to p and whose slope (relative to the coordinate system in $P(p)$) and distance from p are differentiable functions of p. The two corners at the points of intersection of T_1 with T_3 and T_2 with T_3 are "rounded off" and the mapping h is modified in a neighborhood of $f(C)$ to follow the smoothed off curve from $p_1(p)$ to $p_2(p)$.

By a shrinking along interior normals, $h'(M)$ can be chosen to be interior to $h(M)$.

LEMMA 4. *Let $f_1(I)$ and $f_2(I)$ be two regular intervals with $f_1(I) \cup f_2(I) \subset h(M)$ (a regularly imbedded 2-manifold in E^3) and $f_1(I) \cap f_2(I) = f_1(0) = f_2(0)$. Then given $\delta > 0$ there is an interval $f_3(I) \subset h(M)$ such that $f_3(p)$ is within a distance δ of $f_1(I) \cup f_2(I)$ and except for p in the δ-neighborhood of $1/2$, $f_3(p) \in f_1(I) \cup f_2(I)$. Furthermore, we may have the part of $f_3(I)$ not contained in $f_1(I) \cup f_2(I)$ on either "side" of $f_1(I) \cup f_2(I)$ (relative to $h(M)$). Finally, except for the δ-neighborhood of $f_1(0) \cup f_2(0)$, $f_1(I) \cup f_2(I) \subset f_3(I)$.*

The proof of Lemma 4 is the same as that portion of Lemma 3 concerned with the "rounding off" of the intersection of $h(M)$ with $P(p)$. We will not give details here.

Suppose $f(C) \subset M_n$ is a circle and $g(I) \subset M_n$ is an interval such that $g(0) = f(p_1)$, $g(1) = f(p_2)$ and $f(C) \cap g(I) = f(p_1) \cup f(p_2)$. Let A_1 and A_2 be the two intervals of C determined by p_1 and p_2. Then $f(A_1) \cup g(I)$ and $f(A_2) \cup g(I)$ are two topological circles. Given $\delta > 0$ we can alter $f(A_1) \cup g(I)$ (say) in a δ-neighborhood of $f(p_1) \cup f(p_2)$ in accordance with Lemma 4, replacing $f(A_1) \cup g(I)$ by a circle $f'(C)$ which coincides with $f(A_1) \cup g(I)$ except in a δ-neighborhood of $f(p_1) \cup f(p_2)$.

DEFINITION 6. By a δ-alteration of $f(C)$ along A_2 by $g(I)$ we mean a circle $f'(C)$ which is such that

$$[f'(C) - (f(A_2) \cup g(I))] \cup [(f(A_2) \cup g(I)) - f'(C)]$$

is contained in the δ-neighborhood of $f(p_1) \cup f(p_2)$.

5. **Proof of the main result.** The proof of our theorem proceeds by induction on the "complication" of the circles on M_n. As a first step in defining the complication we introduce two circles on T^*. Let ξ be the circle $x = \cos\theta$, $y = \sin\theta$, $z = 0$ $(0 \leqq \theta \leqq 2\pi)$ and let η be the circle $x = -(2 + \cos\phi)$, $y = 0$, $z = \sin\phi$ $(0 \leqq \phi \leqq 2\pi)$. The $3n$ circles $h_i | \xi$, $h_i | \eta$ and $\tilde{g}_i | C$ will be called the basic circles of M_n.

Given a collection of pairwise disjoint circles $\{f_i(C)\}$ $(i = 1, \cdots, m)$ on M_n there is a collection of circles $\{f_i'(C)\}$ isotopic on M_n to $\{f_i(C)\}$ and such that the number of intersections of $\bigcup_{i=1}^{m} f_i'(C)$

with the basic circles is finite (see for example Baer [1, p. 107]).

DEFINITION 7. The complication $K(\{f_i(C)\})$ of a finite collection of pairwise disjoint circles on M_n is the minimum number of intersections of $\{f_i'(C)\}$ with the basic circles of M_n. $\{f_i'(C)\}$ ranges over the class of circles isotopic (on M_n) to $\{f_i(C)\}$.

The theorem will be proved by induction on $K(\{f_i(C)\})$. Suppose first that $K(\{f_i(C)\})=0$. By Lemma 1 we may assume that none of the $\{f_i(C)\}$ intersects any of the basic circles of M_n. This means in particular that each $f_i(C)$ is contained entirely in $e(s)-\bigcup_{i=1}^n \bar{g}_i(D)$ or in some $h_j(T^*)-(h_j(\xi)\cup h_j(\eta))$.

The set $h_j(T^*)\cup g_j(D)$ is a topological torus and by Lemma 3 for any $\delta>0$ there is a regular imbedding $\bar{h}_j(T)$ which coincides with $h_j(T^*)\cup g_j(D)$ except in a δ-neighborhood of $x=3/2$ and in this neighborhood $\bar{h}_j$ is within δ of $\bar{g}_j(C)$. Further, we may assume that $\bar{h}_j(T)$ is contained in the union of $h_j(T^*)\cup g_j(D)$ and its interior.

By choosing δ sufficiently small any given $f_i(C)\subset h_j(T^*)$ will also be such that $f_i(C)\subset\bar{h}_j(T)$. Further, if $f_i(C)\subset\bar{h}_j(T)$ and $f_i(C)\cap(h_j(\xi)\cup h_j(\eta))=\emptyset$ then $f_i(C)\cap(h_j(\xi)\cup h_j(\eta))=\emptyset$. But this last condition implies that $f_i(C)\subset\bar{h}_j(T)-(\bar{h}_j(\xi)\cup\bar{h}_j(\eta))$, an open 2-cell. Therefore $f_i(C)$ is homologous to zero in $\bar{h}_j(T)$.

From these remarks it is evident that our theorem for the case $K(\{f_i(C)\})=0$ is equivalent to the following two propositions:

(I). If $\{f_i(C)\}$ is a pairwise disjoint collection of circles on $e(S)$ (a regularly imbedded sphere in E^3) then there is a pairwise disjoint collection $\{g_i(D)\}$ of discs with $g_i|C=f_i$: C and $g_i(D-C)\subset$interior of $e(S)$.

(II). If $\{f_i(C)\}$ is a pairwise disjoint collection of circles on $h(T)$ (a regularly imbedded torus in E^3) and each $f_i(C)$ is homologous to zero on $h(T)$, there is a pairwise disjoint collection $g_i(D)$ of discs with $g_i|C=f_i$: C and $g_i(D-C)\subset$interior of $h(T)$.

To prove Proposition I we note that if there is just one circle in the collection $\{f_i(C)\}$ then I follows from Lemma 2. Assume I true for collections of circles with fewer than m members and suppose $\{f_i(C)\}$ has m members. Then there is a circle $f_1(C)$ (say) one of whose complementary domains (relative to $e(S)$) contains none of the remaining $f_i(C)$. By the induction hypothesis there is a pairwise disjoint collection of discs $g_2(D), \cdots, g_m(D)$ with $g_i|C=f_i$: C and $g_i(D-C)\subset$interior of $e(S)$. By choosing the δ in Lemma 2 sufficiently small, it is clear there is a disc $g_1(D)$ with $g_1(D-C)\subset$interior of $e(S)$ and with $g_1(D)\cap(\bigcup_{i=2}^m g_i(D))=\emptyset$. This proves I. II can be proved in a similar fashion. This establishes our theorem in the case $K(\{f_i(C)\})=0$.

Assume next the theorem is true for collections of circles with complication $K \leqq k-1$ and suppose $\{f_i(C)\}$ $(i=1, \cdots, m)$ is a collection with complication k. By Lemma 1 we may assume that the number of intersections of $\bigcup_{i=1}^m f_i(C)$ with the basic circles is k. Since $k>0$ we have $(\bigcup_{i=1}^m f_i(C)) \cap (\bigcup_{i=1}^n \tilde{g}_i(C)) \neq \varnothing$. Otherwise, each $f_i(C)$ is contained completely in $e(S)$ or in some $h_t(T^*)$. Suppose $f_i(C)$ is contained in $h_t(T^*)$. Since $f_i(C)$ is homologous to zero in M_n, it separates M_n into two parts, one of which is contained entirely in $h_t(T^*)$. Hence $f_i(C)$ is homologous to zero in $h_t(T^*)$ and the subset of $\bigcup_{i=1}^m f_i(C)$ contained in $h_t(T^*)$ can be brought by an isotopy in $h_t(T^*)$ to a position where it does not intersect $h_t(\xi) \cup h_t(\eta)$. Thus $f_i(C)$ adds nothing to the complication. Therefore, if $K(\{f_i(C)\})>0$, $(\bigcup_{i=1}^m f_i(C)) \cap (\bigcup_{i=1}^n \tilde{g}_i(C)) \neq \varnothing$.

Consider a circle $g_e(C)$ which intersects certain members of $\{f_i(C)\}$ say $f_1(C), \cdots, f_m(C)$ $(0<m' \leqq m)$. Let the points of intersection of $f_i(C)$ $(i=1, \cdots, m')$ with $g_e(C)$ be $p_1^{(i)}, \cdots, p_{k_i}^{(i)}$, numbered in order traversing $g_e(C)$ from a certain fixed point in a certain direction. Between successive pairs $p_r^{(i)}, p_{r+1}^{(i)}$ (or $p_{k_i}^{(i)}, p_1^{(i)}$) there must be an even number of points of each set $p_1^{(j)}, \cdots, p_{k_j}^{(j)}$ $(j \neq i)$. To see this last statement we use

LEMMA 5. *Let $f_1(C)$ and $f_2(C)$ be two disjoint circles on M_n each homologous to zero (in M_n). Let M_1', M_1'' and M_2', M_2'' be respectively the manifolds with boundaries into which $f_1(C)$ and $f_2(C)$ divide M_n. Then with a suitable rearrangement of primes (if necessary) we have $M_1' \subset M_2'$, $M_1'' \supset M_2''$.*

Since $f_1(C)$ and $f_2(C)$ are disjoint, $f_1(C)$ is contained in the interior of (say) M_2' and $f_2(C)$ is contained in the interior of (say) M_1''. Both M_1' and M_1'' cannot meet M_2'' for this would force $f_1(C)$ to intersect $f_2(C)$. From this and the fact that $f_2(C)$ is contained in M_1 it is clear that $M_1' \subset M_2'$. A similar argument shows that $M_2'' \subset M_1''$.

Let M_i', M_i'' be the components into which $f_i(C)$ divides M_n and suppose $f_j(C)$ is contained in M_i''. Then by Lemma 5 a component M_j'' of the complement of $f_j(C)$ is such that $M_j'' \subset M_i''$. Pair together two points of the set $p_1^{(j)}, \cdots, p_{k_j}^{(j)}$ if they can be joined by an interval of $\tilde{g}_e(C)$ lying entirely in M_j. This procedure pairs each point of $p_1^{(j)}, \cdots, p_{k_j}^{(j)}$ with some other point of this collection and the two members of a pair cannot be separated by points of $p_1^{(i)}, \cdots, p_{k_i}^{(i)}$. Hence there must be an even number of points of the set $p_1^{(j)}, \cdots, p_{k_j}^{(j)}$ between successive pairs of $p_1^{(i)}, \cdots, p_{k_i}^{(i)}$.

It is clear from this that for some i and some pair $p_1^{(i)}, p_2^{(i)}$ (say) the interval of $\bar{g}_e(C)$ between $p_1^{(i)}$ and $p_2^{(i)}$ (in the chosen order along $\bar{g}_e(C)$) contains no points of any of the circles $\{f_i(C)\}$.

Let $\bar{g}(I)$ be the interval joining $p_1^{(i)}$ to $p_2^{(i)}$ along $\bar{g}_e(C)$ and which is such that $\bar{g}(I)\cap\bigcup_{j=1}^m f_j(C)=p_1^{(i)}\cup p_2^{(i)}$. The pair $f_i^{-1}(p_1^{(i)})$, $f_i^{-1}(p_2^{(i)})$ divides C into intervals A_1 and A_2. Let $f_i'(C)$ be a δ-alteration of $f_i(C)$ along A_1 by $\bar{g}(I)$ and $f_i''(C)$ be a δ-alteration of $f_i(C)$ along A_2 by $\bar{g}(I)$. The δ-alteration can be chosen so that $(f_i'(C)\cup f_i''(C))\cap\bigcup_{j\neq i} f_j(C)=\emptyset$ and $f_i'(C)\cap f_i''(C)$ is a subinterval of $\bar{g}_e(C)$. By pushing $f_i'(C)$ and $f_i''(C)$ a short distance away from $\bar{g}_e(C)$ in the suitable directions we get two circles $\bar{f}_i'(C)$ and $\bar{f}_i''(C)$ which are disjoint from each other and do not meet the remaining $f_j(C)$.

$\bar{f}_i'(C)$ and $\bar{f}_i''(C)$ each separate M_n so that the set of circles $f_1(C), \cdots, f_{i-1}(C), \bar{f}_i'(C), \bar{f}_i''(C), f_{i+1}(C), \cdots, f_m(C)$ satisfies the conditions of the theorem. Clearly the complication of this set is at least two less than $K(\{f_j(C)\})$. By the induction hypothesis there are discs $\{g_j(D)\}$ $(j\neq i)$, $\bar{g}_i'(D)$, $\bar{g}_i''(D)$ which satisfy the conditions of the theorem with respect to the circles $\{f_j(C)\}$ $(j\neq i)$, $\bar{f}_i'(C)$, $\bar{f}_i''(C)$. By pushing $\bar{g}_i'(D)$ and $\bar{g}_i''(D)$ together along $\bar{g}(I)$ we get a disc $\bar{g}_i(D)$ whose boundary $\bar{g}_i(C)$ is a curve isotopic to $g_i(C)$. $\bar{g}_i(C)$ is straightened out into $g_i(C)$ and $\bar{g}_i(D)$ is deformed into a disc $g_i(D)$ with boundary $g_i(C)$ using Lemma 3. Evidently the deformation can be limited to an arbitrarily small neighborhood of $\bar{g}(I)$ and, therefore, $g_i(D)$ can be chosen so that it does not meet $g_j(D)$ $(j\neq i)$. Also, $g_i(D)$ can be made so that except for $g_i(C)$ it is interior to M_n by using techniques similar to those in Lemma 2.

Therefore, the circles $f_1(C), \cdots, f_m(C)$ bound pairwise disjoint discs $g_1(D), \cdots, g_m(D)$ and the theorem is established by induction.

Without some condition on the imbedding of the surface M_n the theorem above is not true. For example, one can imbed a surface M_2 of genus 2 in 3-space in such a way that there is a circle on M_2 which separates M_2 and does not bound a disc in the interior of M_2.

The theorem can be stated and proved within the framework of piece-wise linear imbeddings instead of regular imbeddings as we have done. This piece-wise linear point of view seems to simplify the proof at certain points and complicate it at others.

References

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