ON UNIONS OF CHAINS OF MODELS

C. C. CHANG

The main purpose of this note is to prove the following theorem in the theory of models: Let K be an arithmetical class (in the wider sense) of relational systems. Then K is the class of all models of some finite or infinite set of sentences of the form

$$(*) \qquad \Pi x_1, \cdots, x_n \Sigma y_1, \cdots, y_m \, \nu$$

where ν is quantifier-free, if and only if K is closed under unions of chains of models.[1] We have adopted most of the notational features of Tarski [7], in which can be found the definitions of relational systems, subsystems, and unions of a chain of systems; we have assumed familiarity with the symbolism of [7]. We have also adopted the convention of giving definitions and theorems only for a similarity class of systems of the type $\langle A, S\rangle$ where S is a ternary relation over A. All the results we have obtained here can be carried over trivially for similarity classes of systems in which there are a finite number of finitary relations.

It will be apparent to the reader that this paper exploits a special method and a special theorem. The method is that of graph-diagrams first introduced in Robinson [6], and the theorem is the completeness theorem of Gödel [3]. We shall use the completeness theorem in the generalized form (by Malcev and Henkin) where we allow an arbitrary number of constants in the first-order predicate calculus. The plan of the paper is first to give a mathematical characterization of those classes K which are determined by sentences of the form $(*)$, then to use this characterization to prove our main result.[2] At the end, we shall apply our result to convex arithmetical classes.

Presented to the Society August 24, 1956; received by the editors February 20, 1958.

[1] This result was announced by the author in [2]. It has previously been announced by Łoś and Suszko in [4] and their joint result recently appeared in [5]. It seems that the two methods of proof are entirely different, and our results (Theorem 7 and Theorem 11) are stronger results than those in [5]. The author takes this opportunity to thank A. Tarski and R. Grewe for pointing out some simplifications in the proofs of Lemmas 8, 9, and 10. These simplifications utilize some results of Tarski and Vaught [9] which were not known to the author.

[2] The characterizations given in Theorems 4, 6, and 7 bear a close analogy to the corresponding characterizations given by Tarski in [7] for universal arithmetical classes.

DEFINITION 1. A class K of relational systems $\langle A, S\rangle$ is an universal-existential class [in the wider sense], in symbols $K\in UEC$ [$K\in UEC_\Delta$], if and only if K is the class of all models of a first-order sentence [a set of first-order sentences] of the form $(*)$.

Let relational systems $\mathfrak{A}=\langle A, S\rangle$, $\mathfrak{A}'=\langle A', S'\rangle$, and $\mathfrak{B}=\langle B, T\rangle$ be given where $\mathfrak{A}'\in S(\mathfrak{A})$, that is, $\mathfrak{A}'$ is a subsystem of $\mathfrak{A}$, and let m be an arbitrary natural number.

DEFINITION 2. $\mathfrak{B}$ is an m-cover of $\mathfrak{A}$ with respect to $\mathfrak{A}'$ if, and only if,

(i) $\mathfrak{A}'$ is isomorphically embeddable in $\mathfrak{B}$ by an isomorphism f, and

(ii) every subsystem $\mathfrak{B}'=\langle B', T'\rangle$ of $\mathfrak{B}$ where $f^*(A')\subseteq B'$, and where the set difference $B'-f^*(A')$ contains no more than m elements, can be isomorphically embedded in $\mathfrak{A}$ by an isomorphism g where g^{-1} restricted to the set A' is equal to f.

Let $\mathfrak{A}'=\langle A', S'\rangle$ be a finite subsystem of $\mathfrak{A}=\langle A, S\rangle$, and let $A'=\{a_1, \cdots, a_n\}$. Consider all possible extensions of A' in A with no more than $n+m$ elements. Each such extension C will give rise to a subsystem $\mathfrak{C}=\langle C, S\cap C^3\rangle$ of $\mathfrak{A}$. Since S is a ternary relation (or, since there are only a finite number of finitary relations), there will exist among all such extensions a finite number of subsystems $\mathfrak{C}_1, \cdots, \mathfrak{C}_r$ with the property that, for any extension $\mathfrak{C}$ of $\mathfrak{A}'$ with no more than $n+m$ elements, there exists a $\mathfrak{C}_i$, $1\leqq i\leqq r$, such that $\mathfrak{C}\cong\mathfrak{C}_i$ and where the isomorphism is the identity on the set A'. Let C_i be the set of elements $\{a_1, \cdots, a_n, b_1, \cdots, b_m\}$ where the b_j's need not be distinct for distinct indices. We introduce a constant c_a for each element $a\in C_i$, and we define the formula $\nu_i(c_{a_1}, \cdots, c_{a_n}, c_{b_1}, \cdots, c_{b_m})$ as follows: Let x_1, x_2, x_3 be elements of C_i. We write

$$\begin{aligned} S(c_{x_1}, c_{x_2}, c_{x_3}) &\text{ if } \langle x_1, x_2, x_3\rangle \in S,\\ \sim S(c_{x_1}, c_{x_2}, c_{x_3}) &\text{ if } \langle x_1, x_2, x_3\rangle \notin S,\\ c_{x_1} \equiv c_{x_2} &\text{ if } x_1 = x_2,\\ c_{x_1} \not\equiv c_{x_2} &\text{ if } x_1 \neq x_2. \end{aligned}$$

The set of all formulas listed on the left constitutes a *description* of $\mathfrak{C}_i$. Since $\mathfrak{C}_i$ is finite, this set of formulas is clearly finite, and we let the formula $\nu_i(c_{a_1}, \cdots, c_{a_n}, c_{b_1}, \cdots, c_{b_m})$ be the conjunction of all formulas in the description of $\mathfrak{C}_i$. Having defined the formulas ν_i, a formula ν depending on the system $\mathfrak{A}$, the subsystem $\mathfrak{A}'$, and the number m is defined by the equality

$$(**)\quad \begin{aligned} \nu(c_{a_1}, \cdots, c_{a_n}) = \Pi y_1, \cdots, y_m\{&\nu_1(c_{a_1}, \cdots, c_{a_n}, y_1, \cdots, y_m)\\ &\vee \cdots \vee \nu_r(c_{a_1}, \cdots, c_{a_n}, y_1, \cdots, y_m)\}. \end{aligned}$$

Finally, we define $\xi = \Sigma x_1, \cdots, x_n \nu(x_1, \cdots, x_n)$. The notation introduced in this paragraph will be used in the proofs of Lemma 3, Theorem 6, and Theorem 11.

LEMMA 3. *Let $\mathfrak{A}'$ be a finite subsystem of $\mathfrak{A}$ and let K be the class of all systems $\mathfrak{B}$ where $\mathfrak{B}$ is not an m-cover of $\mathfrak{A}$ with respect to $\mathfrak{A}'$. Then there exists an $L \in UEC$ such that $K \subseteq L$ and $\mathfrak{A} \notin L$.*

PROOF. Let L be the class of all models of the sentence ξ introduced above. Clearly, $L \in UEC$, $K \subseteq L$, and $\mathfrak{A} \notin L$.

THEOREM 4. *The following two conditions are equivalent*:

(i) $K \in UEC_\Delta$.

(ii) *If for each finite subsystem $\mathfrak{A}'$ of $\mathfrak{A}$ and each m there exists a system $\mathfrak{B} \in K$ where $\mathfrak{B}$ is an m-cover of $\mathfrak{A}$ with respect to $\mathfrak{A}'$, then $\mathfrak{A} \in K$.*

PROOF. Assume (i). Let $\mathfrak{A} = \langle A, S\rangle$ satisfy the hypothesis of (ii) and let $\xi = \Pi x_1, \cdots, x_n \Sigma y_1, \cdots, y_m \nu(x_1, \cdots, x_n, y_1, \cdots, y_m)$ be any one of the characterizing sentences of K. We wish to show that ξ holds in $\mathfrak{A}$. Let any n elements of A be chosen, let $A' = \{a_1, \cdots, a_n\}$ be the set of those elements, and let $\mathfrak{A}' = \langle A', S'\rangle$ be the subsystem determined by A'. Let $\mathfrak{B} = \langle B, T\rangle$ be an m-cover of $\mathfrak{A}$ with respect to $\mathfrak{A}'$ and let $\{f(a_1), \cdots, f(a_n)\} \subseteq B$ be the image of A' under the isomorphism f given by 2(i). Since $\mathfrak{B} \in K$, there exist elements $b_1, \cdots, b_m \in B$ for which

$$\nu(c_{f(a_1)}, \cdots, c_{f(a_n)}, c_{b_1}, \cdots, c_{b_m})$$

holds in $\mathfrak{B}$ under the interpretation of b for c_b for all $b \in B$. Applying now the isomorphism g given by 2(ii), there exist elements $g(b_1), \cdots, g(b_m) \in A$ for which

$$\nu(c_{a_1}, \cdots, c_{a_n}, c_{g(b_1)}, \cdots, c_{g(b_m)})$$

holds in $\mathfrak{A}$ under the interpretation of a for c_a for all $a \in A$. Since the choice of the elements $a_1, \cdots, a_n$ is arbitrary, we see that ξ holds in $\mathfrak{A}$. Hence, $\mathfrak{A} \in K$.

On the other hand, assume that K satisfies 4(ii). Let

$$M = \bigcap \{L; L \in UEC \text{ and } K \subseteq L\}.$$

Clearly $K \subseteq M$ and $M \in UEC_\Delta$. We wish to show the equality $K = M$. Assume $\mathfrak{A} \notin K$, thus, by 4(ii), there exists a finite subsystem $\mathfrak{A}'$ of $\mathfrak{A}$ and an m for which no $\mathfrak{B} \in K$ is an m-cover of $\mathfrak{A}$ with respect to $\mathfrak{A}'$. Obviously, K is included in the class L defined in Lemma 3 and $\mathfrak{A} \notin L$, thus $\mathfrak{A} \notin M$. This proves the inclusion $M \subseteq K$ and hence the equality $K = M$.

We now introduce a notion which is stronger than the notion of a subsystem.

DEFINITION 5. $\mathfrak{A}\in S^*(\mathfrak{B})$ if, and only if, $\mathfrak{A}\in S(\mathfrak{B})$ and every finite subsystem $\mathfrak{B}'$ of $\mathfrak{B}$ can be mapped isomorphically onto a finite subsystem $\mathfrak{A}'$ of $\mathfrak{A}$ by an isomorphism f which is the identity mapping on the set $B'\cap A$.

For a class K, we let $S^*(K)=\{\mathfrak{A};$ there exists $\mathfrak{B}\in K$ such that $\mathfrak{A}\in S^*(\mathfrak{B})\}$.

The following theorem is an analog of a result of Tarski [**7**] on universal arithmetical classes.

THEOREM 6. *The following two conditions are equivalent*:
(i) $K\in UEC_\Delta$.
(ii) $K\in AC_\Delta$ *and* $S^*(K)\subseteq K$.

PROOF. By an argument similar to the proof of 4(ii) from 4(i) we easily obtain 6(ii) from 6(i). Assume now 6(ii). Let $\mathfrak{A}=\langle A, S\rangle$ satisfy the hypothesis of 4(ii), and let Σ be the set of sentences characterizing K. Consider the following enlarged set of sentences Σ' consisting of:

(1) the set Σ,

(2) the set of formulas which forms the description of $\mathfrak{A}$ using the constants c_a for each $a\in A$, and

(3) for each finite subset $\{a_1, \cdots, a_n\}$ of A and each natural number m, the sentence $\nu(c_{a_1}, \cdots, c_{a_n})$ introduced in $(**)$.

We assert that the set Σ' has a model. Any finite subset Σ'' of Σ' will involve at most a finite number of constants $c_{a_1}, \cdots, c_{a_n}$ and a maximal natural number m. Since $\mathfrak{A}$ satisfies the hypothesis of 4(ii), we let $\mathfrak{B}\in K$ be an m-cover of $\mathfrak{A}$ with respect to the system generated by the set $\{a_1, \cdots, a_n\}$. Obviously $\mathfrak{B}$ is a model for Σ''. Hence, by the compactness theorem, there exists a model $\mathfrak{C}$ for Σ'. From the contents of the set Σ', we see that $\mathfrak{C}\in K$ and that $\mathfrak{A}$ is isomorphic to a system $\mathfrak{A}'$ where $\mathfrak{A}'\in S^*(\mathfrak{C})$. By 6(ii), $\mathfrak{A}'\in K$ and, since K is closed under taking isomorphic models, $\mathfrak{A}\in K$. Thus 6(ii) leads to 4(ii), and by Theorem 4, we obtain 6(i).

Just like its analog for universal classes, Theorem 6 can be improved to

THEOREM 7. *If* $K\in AC_\Delta$, *then* $S^*(K)\in UEC_\Delta$.

PROOF. We merely have to show that $S^*(K)$ satisfies condition 4(ii). Suppose $\mathfrak{A}$ is such that for each finite subsystem $\mathfrak{A}'$ of $\mathfrak{A}$ and each m there exists a $\mathfrak{B}\in S^*(K)$ such that $\mathfrak{B}$ is an m-cover of $\mathfrak{A}$ with respect to $\mathfrak{A}'$. Notice that since $\mathfrak{B}\in S^*(K)$, there exists a $\mathfrak{B}'\in K$ such that $\mathfrak{B}\in S^*(\mathfrak{B}')$. It follows easily from Definitions 2 and 5 that $\mathfrak{B}'$

is also an m-cover of $\mathfrak{A}$ with respect to $\mathfrak{A}'$. By the device used in Theorem 6, there exist models $\mathfrak{C}$ and $\mathfrak{D}$ such that $\mathfrak{C}\in K$, $\mathfrak{D}\in S^*(\mathfrak{C})$, and $\mathfrak{A}$ isomorphic to $\mathfrak{D}$. Since K is closed under taking isomorphic models, we conclude that $\mathfrak{A}\in S^*(K)$.

At this point we pause to mention that the following result in the theory of models is implied by Theorem 7, namely: If $K\in AC_\Delta$ and every model of K is finite, then $K\in UEC_\Delta$.[3] Clearly, under the hypothesis that every model of K is finite, $S^*(K)=K$ thus, by Theorem 7, $K\in UEC_\Delta$. By the compactness theorem for arithmetical classes, we see that the subscript Δ may be removed from the above remark. We now proceed to the main result on unions of chains.

For our subsequent discussion we introduce the following notation. Let $\mathfrak{A}\in S^*(\mathfrak{B})$ and let Σ be the set of all sentences constructed with the constants c_a for $a\in A$. We consider a new set of sentences $\Sigma(\mathfrak{A}, \mathfrak{B})$ with constants c_b for $b\in B$ consisting of:

(1) the set of all sentences of Σ which hold in $\mathfrak{A}$ under the interpretation of a for c_a, for all $a\in A$, and

(2) the description of $\mathfrak{B}$.

We are now ready to prove the following lemmas.

LEMMA 8. $\Sigma(\mathfrak{A}, \mathfrak{B})$ *has a model.*

PROOF. Any finite subset Σ' of $\Sigma(\mathfrak{A}, \mathfrak{B})$ will contain only a finite number of constants c_b for $b\in B$. Let B' be the finite subset of B determined by the constants occurring in Σ' and let $\mathfrak{B}'$ be the corresponding subsystem. By Definition 5, there exists a subsystem $\mathfrak{A}'$ of $\mathfrak{A}$ which is isomorphic to $\mathfrak{B}'$ and such that the isomorphism f is the identity mapping on the set $B'\cap A$. Clearly, $\mathfrak{A}$ is a model for Σ' if each constant c_b for $b\in B'$ is interpreted as the element $f(b)\in A$. Thus, by the compactness theorem, $\Sigma(\mathfrak{A}, \mathfrak{B})$ has a model.

LEMMA 9. *Let* $K\in AC_\Delta$, $\mathfrak{C}_0\in K$, $\mathfrak{A}_0\in S^*(\mathfrak{C}_0)$, *and* $\Sigma_0=\Sigma(\mathfrak{A}_0, \mathfrak{C}_0)$. *Then there exist an infinite sequence of models* $\mathfrak{A}_0$, $\mathfrak{C}_0$, $\mathfrak{A}_1$, $\mathfrak{C}_1, \cdots$, $\mathfrak{A}_n$, $\mathfrak{C}_n, \cdots$ *and an infinite sequence of sets of sentences* $\Sigma_0, \Sigma_1, \cdots$, $\Sigma_n, \cdots$ *such that for each natural number n the following hold*:

(i) $\mathfrak{A}_n\in S^*(\mathfrak{C}_n)$.

(ii) $\mathfrak{C}_n\in S(\mathfrak{A}_{n+1})$ *and* $\mathfrak{C}_n\in K$.

(iii) $\Sigma_n=\Sigma(\mathfrak{A}_n, \mathfrak{C}_n)$.

(iv) *Every sentence of* Σ_n *holds in* $\mathfrak{A}_{n+1}$.

(v) *If* $m\leq n$, *then* $\mathfrak{A}_n$ *is an arithmetical extension of* $\mathfrak{A}_m$.[4]

PROOF. The lemma is proved by mathematical induction. Assume we already have the models $\mathfrak{A}_0$, $\mathfrak{C}_0, \cdots$, $\mathfrak{A}_n$, $\mathfrak{C}_n$, and the sets of

[3] This result was known to Tarski and was orally communicated to the author.

[4] For the notion of arithmetical extensions, cf. Definition 1.6 in [9].

sentences $\Sigma_0, \cdots, \Sigma_n$. We can now obtain the model $\mathfrak{A}_{n+1}$ with the aid of Lemma 8, the inductive hypothesis, and the fact that K is closed under taking isomorphic models. In order to prove (v), by Theorem 1.8(ii) of [9] it is sufficient to show that $\mathfrak{A}_{n+1}$ is an arithmetical extension of $\mathfrak{A}_n$; i.e., that

(1) every sentence constructed with the constants c_a, for $a \in A_n$, which holds in $\mathfrak{A}_n$ also holds in $\mathfrak{A}_{n+1}$.

Condition (1), of course, follows immediately from (iii) and (iv). From (v) and Corollary 1.7 of [9], we see that $\mathfrak{A}_0$ and $\mathfrak{A}_{n+1}$ are arithmetically equivalent. Since $\mathfrak{A}_0 \in S^*(K)$ and $S^*(K) \in UEC_\Delta$, therefore $\mathfrak{A}_{n+1} \in S^*(K)$. We now simply let $\mathfrak{C}_{n+1}$ be such a model that $\mathfrak{C}_{n+1} \in K$ and $\mathfrak{A}_{n+1} \in S^*(\mathfrak{C}_{n+1})$. Obviously, $\mathfrak{C}_n \in S(\mathfrak{A}_{n+1})$. The induction is complete and the lemma is proved.

Lemma 10. *Given the sequence of models as in Lemma* 9, $\mathfrak{A}_0$ *and* $\bigcup \mathfrak{A}_n$ *are arithmetically equivalent.*

Proof. By Theorem 1.9 of [9], $\bigcup \mathfrak{A}_n$ is an arithmetical extension of $\mathfrak{A}_0$. Thus, by Corollary 1.7 of [9], the conclusion of the lemma follows.

Given a class K we let $CL(K) = \{\mathfrak{A};$ there exists a $\mathfrak{B} \in K$ such that $\mathfrak{A}$ is arithmetically equivalent with $\mathfrak{B}\}$.

Theorem 11. *If* $K \in AC_\Delta$, *then* $S^*(K) = CL(\bigcup(K))$.

Proof. In order to prove the inclusion $CL(\bigcup(K)) \subseteq S^*(K)$ it is sufficient, by Theorem 7, to prove that $\bigcup(K) \subseteq S^*(K)$. To this end, let $\mathfrak{A}$ be the union of a chain of models $\mathfrak{A}_i \in K$, with $i \in I$. We consider the following set of sentences Σ' consisting of:

(1) The set Σ of sentences characterizing K.

(2) The description of $\mathfrak{A}$.

(3) For each finite subset $\{a_1, \cdots, a_n\}$ of A and each natural number m, the sentence $\nu(c_{a_1}, \cdots, c_{a_n})$ introduced in $(**)$.

Any finite subset Σ'' of Σ' will contain only a finite number of constants c_a for $a \in A$. Any finite number of elements of A belongs to A_i for some $i \in I$. It is obvious that $\mathfrak{A}_i$ will be a model for Σ''. Thus, by the compactness theorem, there exists a model $\mathfrak{B}$ for Σ'. We see from the construction of Σ' that $\mathfrak{B} \in K$ and that there exists a model $\mathfrak{A}'$ such that $\mathfrak{A}' \in S^*(\mathfrak{B})$ and $\mathfrak{A}$ isomorphic to $\mathfrak{A}'$. Thus $\mathfrak{A} \in S^*(K)$.

In order to prove the inclusion $S^*(K) \subseteq CL(\bigcup(K))$, let $\mathfrak{A} \in S^*(\mathfrak{C})$ and $\mathfrak{C} \in K$. Letting $\mathfrak{A} = \mathfrak{A}_0$ and $\mathfrak{C} = \mathfrak{C}_0$, we repeat the construction of the sequences $\mathfrak{A}_0, \mathfrak{C}_0, \cdots, \mathfrak{A}_n, \mathfrak{C}_n, \cdots$ and the sets $\Sigma_0, \cdots, \Sigma_n, \cdots$ as in Lemma 9. It is clear that $\bigcup \mathfrak{A}_n = \bigcup \mathfrak{C}_n$. Since $\mathfrak{C}_n \in K$ for each n, $\bigcup \mathfrak{C}_n \in \bigcup(K)$ and hence $\bigcup \mathfrak{A}_n \in \bigcup(K)$. By Lemma 10, $\mathfrak{A}$ and

$\bigcup \mathfrak{A}_n$ are arithmetically equivalent, thus $\mathfrak{A} \in CL(\bigcup(K))$. The theorem is proved.

THEOREM 12. *If $K \in AC_\Delta$, then $K \in UEC_\Delta$ if, and only if, $\bigcup(K) \subseteq K$.*

PROOF. Assume $K \in UEC_\Delta$. By Theorem 6, $S^*(K) \subseteq K$ and, by Theorem 11, $\bigcup(K) \subseteq K$. On the other hand, if $\bigcup(K) \subseteq K$, then, by Theorem 11, $CL(\bigcup(K)) = CL(K) = K = S^*(K)$, thus $S^*(K) \subseteq K$. By Theorem 6 again, $K \in UEC_\Delta$. The theorem is proved.

It is now of some interest to ask whether $K \in AC_\Delta$ implies $\bigcup(K) \in UEC_\Delta$. In what follows we exhibit a $K \in AC$ such that $\bigcup(K) \notin UEC_\Delta$.

We consider relational systems $\langle A, R\rangle$ formed by a nonempty set A and a binary relation R over A. Let K be the arithmetical class of models $\langle A, R\rangle$ determined by the conditions: the relation R is a simply ordering relation on the elements of A, if an element of A has an immediate predecessor then it also has an immediate successor, and there exists an element of A which is the successor of every element of A. Let ordinals be defined in such a way that each ordinal is the set of all smaller ordinals, and let $\leqq$ be the well-ordering relation among ordinals. If we let ω denote the least infinite ordinal, then the following will hold:

$$\langle \omega, \leqq\rangle \in S^*(\langle \omega + 1, \leqq\rangle), \qquad \langle \omega + 1, \leqq\rangle \in K,$$

and

$$\langle \omega, \leqq\rangle \in S^*(K).$$

Since no finite subsystem of $\langle \omega, \leqq\rangle$ with more than one element belongs to K, and no infinite subsystem of $\langle \omega, \leqq\rangle$ belongs to K, we see that $\langle \omega, \leqq\rangle \notin \bigcup(K)$. If $\bigcup(K) \in UEC_\Delta$, then by Theorem 11, $S^*(K) = \bigcup(K)$. Thus $\langle \omega, \leqq\rangle \in \bigcup(K)$ which is a contradiction.

In the concluding part of the paper we apply our results to those arithmetical classes which are convex.[5]

DEFINITION 13. $K \in AC_\Delta$ is convex if, and only if, whenever $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K$, $\mathfrak{B}, \mathfrak{C} \in S(\mathfrak{A})$, and $\mathfrak{B} \cap \mathfrak{C}$ is not empty, then $\mathfrak{B} \cap \mathfrak{C} \in K$.

THEOREM 14. *If $K \in AC_\Delta$ and K is convex, then $\bigcup(K) \subseteq K$.*

PROOF. Let $\mathfrak{A}_i$, $i \in I$, be a chain of models of K, and let $\mathfrak{A} = \bigcup \mathfrak{A}_i$. By

[5] The definition of a "convex algebra of axioms" is found in Robinson [6]. Stated in terms of arithmetical classes, his notion is much stronger than what we have assumed here. Thus, our Theorem 14 is essentially a strengthening of (9.1.3) of [6]. Theorem 14 was announced by the author in [1]; we point out here that all other results announced in [1] can be proved in a manner similar to the proof of Theorem 14.

the compactness theorem and the fact that $K \in AC_\Delta$, we see that there exists a model $\mathfrak{B} \in K$ such that $\mathfrak{A} \in S(\mathfrak{B})$. Let us now introduce constants c_b for each $b \in B$ and an unary predicate symbol P. We consider the set of sentences Σ' consisting of:

(1) The set Σ of sentences characterizing K.

(2) The set $\Sigma^{(P)}$ obtained by relativizing all sentences of Σ to P.[6]

(3) The set of all sentences $P(c_a)$ for all $a \in A$.

(4) The set of all sentences $\sim P(c_b)$ for all $b \in B-A$.

(5) The description of $\mathfrak{B}$.

Every finite subset Σ'' of Σ' will involve at most a finite number of constants. For some $\mathfrak{A}_j$, the set A_j will contain all those elements corresponding to those constants in Σ''. Clearly, $\mathfrak{B}$ will be a model for Σ'' if the unary predicate is interpreted as the set A_j. Hence, by the compactness theorem, there exists a model $\mathfrak{C}$ for Σ'. Let D be the set of elements of C which is the interpretation of P in C, and let $\mathfrak{D}$ be the corresponding subsystem of $\mathfrak{C}$. By (2), $\mathfrak{D} \in K$. Let A' be the set of elements of C which are interpretations of the constants c_a, for $a \in A$, and let $\mathfrak{A}'$ be the corresponding subsystem. Similarly, let B' be the set of elements of C which are interpretations of the constants c_b, for $b \in B$, and let $\mathfrak{B}'$ be the corresponding subsystem. We see that $\mathfrak{A} \cong \mathfrak{A}'$, $\mathfrak{B} \cong \mathfrak{B}'$, and $\mathfrak{B}' \cap \mathfrak{D} = \mathfrak{A}'$. Thus $\mathfrak{A}' \in K$, and, finally, $\mathfrak{A} \in K$. The theorem is proved.

COROLLARY 15. *If $K \in AC_\Delta$ and K is convex, then $K \in UEC_\Delta$.*

REFERENCES

1. C. C. Chang, *A remark on convex classes*, Bull. Amer. Math. Soc. Abstract 60-4-579.

2. ———, *On the union of chains of models*, Bull. Amer. Math. Soc. Abstract 62-6-775.

3. K. Gödel, *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, Monatshefte für Mathematik und Physik, vol. 37 (1930) pp. 349–360.

4. J. Łoś and R. Suszko. *On the infinite sums of models*, Bull. Acad. Polon. Sci. Cl. III vol. 3 (1955) pp. 201–202.

5. ———, *On the extending of models*, IV, Fund. Math. vol. 44 (1957) pp. 52–60.

6. A. Robinson, *On the metamathematics of algebra*, Amsterdam, 1951.

7. A. Tarski, *Contributions to the theory of models*, Parts I and II, Indag. Math. vol. 16 (1954) pp. 572–588.

8. A. Tarski, A. Mostowski, and R. M. Robinson, *Undecidable theories*, Amsterdam, 1953.

9. A. Tarski and R. L. Vaught, *Arithmetical extensions of relational systems*, Compositio Math. vol. 13 (1957) pp. 81–102.

UNIVERSITY OF CALIFORNIA, LOS ANGELES

[6] For the notion of relativizing a sentence to an unary predicate P, cf. [8, pp. 24–25].