ON ALMOST PERIODIC SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS

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Let C^n denote n-dimensional complex vector space with vectors x whose components with respect to a given basis are denoted by $x=(x^i)$. Let $\|x\|$ denote the norm of x defined by the equation

$$\|x\| = \sum_{i=1}^{n} |x^i|.$$

Then C^n is a metric space (C^n, d) with metric d defined by the equation

$$d(x, y) = \|x - y\|.$$

A set E of real numbers is called *relatively dense* if there exists a positive real number l such that every interval of length l contains at least one member of the set E.

Following Tornehave [4], a continuous function $x(t)$, defined for all real t and having values in a metric space (X, d), is called an *almost periodic movement* in X if to each positive real number ϵ there corresponds a relatively dense set of real numbers τ such that

$$d(x(t + \tau), x(t)) \leqq \epsilon$$

for all real t.

Similarly, a function $x(k)$, defined on the integers $k=0, \pm 1, \pm 2, \cdots$ and having values in a metric space (X, d), will be called an *almost periodic sequence* in X if to each positive real number ϵ there corresponds a relatively dense set of integers τ such that

$$d(x(k + \tau), x(k)) \leqq \epsilon$$

for $k=0, \pm 1, \pm 2, \cdots$. In case (X, d) is the space of complex numbers this corresponds to the almost periodic sequences studied by Ingeborg Seynsche [3].

Consider the vector differential equation

$$(1) \qquad dx/dt = F(t, x) \qquad -\infty < t < +\infty,$$

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where

(a) $F(t, x)$ is an almost periodic movement in C^n for each x in some connected open subset D of C^n, and

(b) $F(t, x)$ satisfies the following Lipschitz condition for all x and y in D:

$$\|F(t, x) - F(t, y)\| \leqq K\|x - y\| \qquad \text{for all real } t.$$

We shall give a necessary and sufficient condition that a given solution $\phi(t)$ of (1) is almost periodic (=almost periodic movement in C^n).

Now $F(t, x)$ can be thought of as a family of almost periodic movements in C^n. For this purpose we shall adopt Tornehave's definition [**4**, Definition 2, p. 5] of a uniformly continuous family of almost periodic movements. Let S denote a compact metric space and let (X, d) denote an arbitrary metric space with metric d. A function $f(t; v)$ defined for all real t and all v in S, and with values in X, is called a *uniformly continuous family of almost periodic movements* in X if:

(a) The function $f(t; v)$ is an almost periodic movement for each v in S, and

(b) To each positive real ϵ and each v_0 in S there corresponds a neighborhood $U(\epsilon; v_0)$ of v_0 such that $d(f(t; v), f(t; v_0)) \leqq \epsilon$ for all real t and all v in $U(\epsilon; v_0)$.

LEMMA 1. *Let C denote a compact subset of D, the connected open subset of C^n on which $F(t, x)$ is defined. Then $F(t, x)$ with x restricted to C is a uniformly continuous family of almost periodic movements in C^n.*

PROOF. Condition (a) of the definition follows immediately from condition (a) of Differential Equation (1). To each positive real number ϵ and each x_0 in C choose

$$U(\epsilon; x_0) = \{x: \|x - x_0\| \leqq \epsilon/K\}.$$

Then by condition (b) of the Differential Equation (1) we have

$$\|F(t, x) - F(t, x_0)\| \leqq K\|x - x_0\| \leqq \epsilon$$

for all x in $U(\epsilon; x_0)$ and all real t. Hence condition (b) of the definition is satisfied.

LEMMA 2. *If $x(k)$ is an almost periodic sequence in C^n, then for each positive real number ϵ the set T of ϵ-translation integers common to $x(k)$ and all the functions of the family $\{F(t, x): x \in C \subset D\}$ is a relatively dense set.*

PROOF. Hans Tornehave [**4**, Theorem 1] has shown that to each uniformly continuous family $f(t; v)$ of almost periodic movements in a metric space (X, d) there corresponds a real-valued almost periodic function $g(t)$ such that, for each positive ϵ, the set of ϵ-translation numbers common to all the functions of the family $f(t; v)$ contains the set of ϵ-translation numbers of $g(t)$. Hence to prove the lemma, we have only to show that if $x(k)$ is an almost periodic sequence in C^n and $g(t)$ is a real-valued almost periodic function, then for each positive ϵ the set of ϵ-translation integers common to $x(k)$ and $g(t)$ is a relatively dense set. The proof of this is analogous to the proof that the sum of two almost periodic sequences is again an almost periodic sequence. (Cf. [**3**]). However, for the reader's convenience, a detailed proof is given here.

Consider S. Bochner's *translation function* $e(k)$, $k=0, \pm 1, \pm 2, \cdots$, (cf. [**1**, pp. 8–9]) defined by the equation

$$e(k) = \sup \{ \| x(j+k) - x(j) \| : j = 0, \pm 1, \pm 2, \cdots \}.$$

The following three properties of this function are easy consequences of its definition:

(a) $e(k) \geqq 0$, $e(0)=0$,

(b) $e(-k)=e(k)$,

(c) $e(j+k) \leqq e(j)+e(k)$.

Furthermore $e(k)$ is an almost periodic sequence. To prove this we write

$$e(k+\tau) \leqq e(k) + e(\tau),$$

and

$$e(k) \leqq e(k+\tau) + e(-\tau) = e(k+\tau) + e(\tau),$$

and hence

$$| e(k+\tau) - e(k) | \leqq e(\tau) \qquad \text{for all integers } k \text{ and } \tau.$$

For $k=0$ we have

$$| e(\tau) - e(0) | = e(\tau),$$

and so

$$\sup_k | e(k+\tau) - e(k) | = e(\tau).$$

That is,

$$\sup_k | e(k+\tau) - e(k) | = \sup_j \| x(j+\tau) - x(j) \|.$$

From this last equation we conclude that the set of ϵ-translation integers τ of $e(k)$ is identical to the set of ϵ-translation integers of $x(k)$. Since the last mentioned set is relatively dense for each positive ϵ, $e(k)$ is an almost periodic sequence.

Thus to prove the lemma, we need only to show that the set of ϵ-translation integers common to $g(t)$ and $e(k)$ is a relatively dense set. To this end let τ_g and τ_e denote translation integers of $g(t)$ and $e(k)$, respectively. It is well-known that every almost periodic function possesses a relatively dense set of ϵ-translation integers for each positive real number ϵ. (Cf. [1, Paragraph 4°, p. 54]). Let l_0 be a positive integer such that every interval $[k, k+l_0]$, $k=0, \pm 1, \pm 2, \cdots$, contains an $\epsilon/2$-translation integer τ_g and an $\epsilon/2$-translation integer τ_e. Let F_k denote the interval $[kl_0, kl_0+l_0]$ for $k=0, \pm 1, \pm 2, \cdots$. In each F_k select two $\epsilon/2$-translation integers, $\tau_e^{(k)}$ and $\tau_g^{(k)}$. The differences $d^{(k)}=\tau_e^{(k)}-\tau_g^{(k)}$ satisfy

$$-l_0 \leqq d^{(k)} \leqq l_0 \text{ for every integer } k.$$

Those of the $2l_0+1$ integers $-l_0, \cdots, l_0$ which occur among the differences $d^{(k)}$ for some k we shall denote by $\{i_1, i_2, \cdots, i_p\}$, $1 \leqq p \leqq 2l_0+1$. For each i_ν, $\nu=1, 2, \cdots, p$, there exists an integer r_ν with smallest absolute value $|r_\nu|$ such that $d^{(r_\nu)}=i_\nu$. Let

$$R = \max \{ |r_\nu| : \nu = 1, 2, \cdots, p\}.$$

Choose an integer $L > l_0$ so large that the interval $[-L, L]$ contains the interval F_R. Since $L > l_0$, the interval $[m-L, m+L]$ contains at least one of the intervals F_k (of length l_0), say F_r. Select from the intervals F_k contained in the interval $[-L, L]$ the one interval F_q for which $d^{(q)}=d^{(r)}$. Then

$$(\tau_e^{(r)} - \tau_e^{(q)}) - (\tau_g^{(r)} - \tau_g^{(q)}) = d^{(r)} - d^{(q)} = 0.$$

Hence, since the sum (or difference) of $\epsilon/2$-translation numbers is an ϵ-translation number, if we set $\tau'=\tau_e^{(r)}-\tau_e^{(q)}=\tau_g^{(r)}-\tau_g^{(q)}$ then τ' is a common ϵ-translation integer of $e(k)$ and $g(t)$, and since $-L \leqq \tau_e^{(q)} \leqq L$ and $m-L \leqq \tau_e^{(r)} \leqq m+L$ we have

$$m - 2L \leqq \tau' \leqq m + 2L.$$

Since m is an arbitrary integer the last inequality proves that the numbers τ' are relatively dense. This completes the proof of Lemma 2.

THEOREM. *Let $\phi(t)$ be a vector solution of Differential Equation* (1) *for all real t and let D contain the closure of the range of $\phi(t)$. Then a necessary and sufficient condition that $\phi(t)$ be an almost periodic movement in C^n is that $\phi(k)$, $k=0, \pm 1, \pm 2, \cdots$, be an almost periodic sequence in C^n.*

PROOF. The necessity is trivial since every almost periodic function is an almost periodic sequence when restricted to the integers. This fact is an immediate consequence of [1, Paragraph 4°, p. 54] and. the fact that the set of ϵ-translation numbers of $\phi(t)$ is identical to the set of ϵ-translation numbers of Bochner's translation function

$$e(t) = \sup_s \|\phi(s + t) - \phi(s)\|. \qquad \text{(Cf.[1, pp. 8–9]).}$$

Thus it is the sufficiency of the condition with which we are primarily concerned here. Consequently for the remainder of the proof we assume that $\phi(k)$ is an almost periodic sequence in C^n and that $\phi(t)$ satisfies Differential Equation (1) for all real t. With these hypotheses we intend to show that $\phi(t)$ is an almost periodic movement in C^n.

First of all we shall prove that the range of $\phi(t)$ is a bounded set in C^n. Clearly

$$\|\phi(t)\| \leqq \|\phi(t) - \phi(k)\| + \sup_i \|\phi(i)\|, \tag{2}$$

where $\sup_i \|\phi(i)\|$ is finite since every almost periodic sequence is bounded. Moreover

$$\begin{aligned} \|F(t, \phi(k))\| &\leqq \|F(t, \phi(k)) - F(t, \phi(0))\| + \|F(t, \phi(0))\| \\ &\leqq K \sup_k \|\phi(k) - \phi(0)\| + \sup_t \|F(t, \phi(0))\|. \end{aligned} \tag{3}$$

Thus $F(t, \phi(k))$ is bounded for all integers k and all real t. Let M denote $\sup_{k,t} \|F(t, \phi(k))\|$. Since the Differential Equation (1) can be written as the integral equation

$$\phi(t) = \phi(0) + \int_0^t F(s, \phi(s))ds$$

we can use (3) and the Lipschitz condition on $F(t, x)$ to obtain the following inequality which holds for all real t in the interval $k \leqq t \leqq k+1$.

$$\begin{aligned} \|\phi(t) - \phi(k)\| &\leqq \left\| \int_k^t F(s, \phi(s))ds - \int_k^t F(s, \phi(k))ds \right\| \\ &\quad + \left\| \int_k^t F(s, \phi(k))ds \right\| \\ &\leqq K \int_k^t \|\phi(s) - \phi(k)\| ds + M. \end{aligned} \tag{4}$$

From (4) it follows (cf. [2, Chapter 1, Problem 1]) that

$$\|\phi(t) - \phi(k)\| \leqq Me^{K} \qquad \text{for } k \leqq t \leqq k+1. \tag{5}$$

Combining (2) and (5) we now have

$$\|\phi(t)\| \leqq Me^{K} + \sup_{i} \|\phi(i)\| \qquad k \leqq t \leqq k+1$$

for each integer k. Thus $\phi(t)$ is a bounded function.

Now in C^n the closure of a bounded set is bounded and since compact sets in C^n are precisely those sets which are both closed and bounded it follows from Lemma 1 that $F(t, x)$, with x restricted to the closure of the range of $\phi(t)$, is a uniformly continuous family of almost periodic movements in C^n. The range of $\phi(t)$ will be denoted by R, and its compact closure by $\overline{R}$. Now by Lemma 2 there exists, for each positive real number ϵ, a relatively dense set T of ϵ-translation integers τ common to $\phi(k)$ and all the functions of the family $\{F(t, x): x \in \overline{R}\}$. We are now in a position to show that $\phi(t)$ is an almost periodic movement in C^n.

$\phi(t)$ is certainly continuous for all t since we have assumed it to be differentiable for all t. Thus it only remains to be shown that $\phi(t)$ possesses a relatively dense set of ϵ-translation numbers corresponding to each positive ϵ. Consider the following inequality.

$$\begin{aligned}
\|\phi(t+\tau) - \phi(t)\| &= \left\| \left(\int_0^{t+\tau} - \int_0^{t} \right) F(s, \phi(s))ds \right\| \\
&= \left\| \left(\int_0^{k+\tau} - \int_0^{k} + \int_{k+\tau}^{t+\tau} - \int_k^{t} \right) F(s, \phi(s))ds \right\| \\
&\leqq \|\phi(k+\tau) - \phi(k)\| \\
&\quad + \int_k^t \|F(s+\tau, \phi(s+\tau)) - F(s, \phi(s+\tau))\| ds \\
&\quad + \int_k^t \|F(s, \phi(s+\tau)) - F(s, \phi(s))\| ds.
\end{aligned}$$

If we now let τ be an ϵ-translation integer of the relatively dense set T of ϵ-translation integers common to $\phi(k)$ and all $F(t, x)$ for x in $\overline{R}$, the last inequality becomes

$$\begin{aligned}
\|\phi(t+\tau) - \phi(t)\| &\leqq \epsilon + \epsilon(t-k) + K \int_k^t \|\phi(s+\tau) - \phi(s)\| ds \\
&\leqq 2\epsilon + K \int_k^t \|\phi(s+\tau) - \phi(s)\| ds,
\end{aligned}$$

for all t in the interval $k \leqq t \leqq k+1$. But then, as before, it follows (cf. [2, Chapter 1, Problem 1]) that

$$\|\phi(t+\tau) - \phi(t)\| \leqq 2\epsilon e^K \qquad k \leqq t \leqq k+1,$$

for every integer k. Thus we see that an ϵ-translation integer common to $\phi(k)$ and all $F(t, x)$ for x in $\overline{R}$ is a $2\epsilon e^K$-translation integer for $\phi(t)$. Therefore $\phi(t)$ is an almost periodic movement in C^n. This completes the proof of the theorem.

References

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