CONVERGENCE OF CONTINUED FRACTIONS OF STIELTJES TYPE

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1. **Introduction.** In this paper we consider the continued fraction

$$(1.1)\qquad \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots,$$

where $b = \{b_p\}_{p=1}^{\infty}$ is a complex number sequence. If $\{g_p\}_{p=0}^{\infty}$ is the sequence of approximants of (1.1), then $g_p = C_p/D_p$, where

$$(1.2)\qquad \begin{array}{llll} C_0 = 0, & C_1 = 1, & C_{q+1} = b_{q+1}C_q + C_{q-1}, & \\ D_0 = 1, & D_1 = b_1, & D_{q+1} = b_{q+1}D_q + D_{q-1}. & \end{array} \quad q = 1, 2, 3, \cdots.$$

One result presented here is an extension of a result mentioned in [**3**].[1] In [**3**] it is stated that if $\{g_{2p-1}\}(\{g_{2p}\})$ converges absolutely and $\{g_{2p}\}(\{g_{2p-1}\})$ converges, then (1.1) converges if the series $\sum |b_{2p-1}|$ (the series $\sum |b_{2p}|$) diverges. The extension of this result is that if $\{g_{2p-1}\}(\{g_{2p}\})$ converges absolutely and $\{g_{2p}\}(\{g_{2p-1}\})$ converges, then (1.1) converges if, and only if, b satisfies one of the following conditions:

(1) the series $\sum |b_{2p-1}|$ (the series $\sum |b_{2p}|$) diverges,
(2) $\limsup |b_2 + b_4 + \cdots + b_{2p}| = \infty$, ($\limsup |b_1 + b_3 + \cdots + b_{2p-1}| = \infty$).

The statement that b satisfies condition (H) [**2**, p. 122] means one of the following three statements holds:

(1) the series $\sum |b_{2p-1}|$ diverges,
(2) the series $\sum |b_{2p+1}(b_2 + b_4 + \cdots + b_{2p})^2|$ diverges,
(3) $\lim |b_2 + b_4 + \cdots + b_{2p}| = \infty$.

Another result given here is that if $\sum |b_{2p+1}| \, |D_{2p}|^2$ converges and $\{g_p\}_{p=k}^{\infty}$ is a bounded complex number sequence for some k, then (1.1) converges if, and only if, b satisfies (3) of condition (H). Finally we use this result to obtain a very simple proof of a theorem of Scott and Wall [**1**]: Let $k_1, k_2, k_3, \cdots$ be constants such that $k_1 > 0$, $k_{2p+1} \geqq 0$, $\operatorname{Re}(k_{2p}) \geqq 0$, $p = 1, 2, 3, \cdots$, and let $z_1, z_2, z_3, \cdots$ be complex variables. The continued fraction

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[1] Numbers in brackets refer to references at the end of the paper.

$$\frac{1}{k_1 z_1} + \frac{1}{k_2} + \frac{1}{k_3 z_2} + \frac{1}{k_4} + \cdots \tag{1.3}$$

converges for Re $(z_p) \geqq d$, $|z_p| < M$, $p = 1, 2, 3, \cdots$, where d and M are any positive constants, if, and only if, the sequence $k_1, k_2, k_3, \cdots$ satisfies condition (H). This theorem includes an important convergence theorem of Van Vleck [**2**, p. 131], a result of Hamburger [**2**, p. 133], and an extension of Hamburger's theorem due to Mall [**2**, p. 132].

2. **Results.** Using summation by parts, we note that if k is a positive integer such that $D_{2q-1} \neq 0$ for $q \geqq k$, then if $n > k$,

$$\begin{aligned} &\sum_{p=k}^{n} (D_{2p} - D_{2p-2}) \frac{1}{D_{2p-1}} \\ &\quad = D_{2n}/D_{2n-1} - D_{2k-2}/D_{2k-1} + \sum_{p=k}^{n-1} D_{2p}(1/D_{2p-1} - 1/D_{2p+1}). \end{aligned} \tag{2.1}$$

This method of summation will play an important part in the proofs of our theorems.

In order to establish the result stated in [**3**] which we mentioned in the introduction, we proved a lemma to the effect that if z is a complex number sequence such that the series $\sum |1 - z_{p+1}/z_p|$ converges, then z converges absolutely. We now wish to strengthen this lemma by adding to the conclusion that z does not converge to zero. Suppose lim $z_p = 0$. Let $|1 - z_{p+1}/z_p| = r_p$, $p = 1, 2, 3, \cdots$. Let j be a positive integer such that $\sum_{(p \geqq j)} r_p < 1$. Let t be a positive integer greater than j such that $|z_t| \geqq |z_p|$ for $p \geqq t$. Then, since

$$|z_p| - |z_{p+1}| \leqq |z_p - z_{p+1}| = r_p |z_p|,$$

we see that

$$|z_t| - |z_{t+n}| \leqq \sum_{p=0}^{n-1} r_{t+p} |z_{t+p}| \leqq |z_t| \sum_{p=0}^{n-1} r_{t+p},$$

and so

$$|z_t| \left[1 - \sum_{p=0}^{n-1} r_{t+p} \right] \leqq |z_{t+n}|.$$

Hence we have a contradiction, and lim $z_p \neq 0$.

THEOREM A. *If* $\{g_{2p-1}\}$ $(\{g_{2p}\})$ *converges absolutely and* $\{g_{2p}\}$ $(\{g_{2p-1}\})$ *converges, then the continued fraction* (1.1) *converges if, and only if,* b *satisfies one of the following conditions*:

(1) *the series* $\sum|b_{2p-1}|$ (*the series* $\sum|b_{2p}|$) *diverges,*
(2) $\limsup |b_2+b_4+\cdots+b_{2p}|=\infty$ ($\limsup |b_1+b_3+\cdots+b_{2p-1}|=\infty$).

PROOF. We note that if b satisfies condition (H), or condition (H) with even and odd subscripts interchanged, then (1) or (2) must hold. Scott and Wall [1] showed that for (1.1) to converge it is necessary that b satisfy condition (H). Hence we need only consider the proof of the sufficiency. Suppose $\{g_{2p-1}\}$ converges absolutely and $\{g_{2p}\}$ converges, but (1.1) does not converge. Let k be a positive integer such that $D_{2q}\neq 0$, $D_{2q-1}\neq 0$ for $q\geqq k$. Then if $p\geqq k$,

$$(2.2)\qquad \begin{aligned} |g_{2p+1}-g_{2p-1}| &= \frac{|b_{2p+1}|}{|D_{2p+1}D_{2p-1}|} = \frac{|b_{2p+1}|\,|D_{2p}|^2}{|D_{2p+1}D_{2p}|\,|D_{2p}D_{2p-1}|} \\ &= |g_{2p+1}-g_{2p}|\,|g_{2p}-g_{2p-1}|\,|b_{2p+1}|\,|D_{2p}|^2. \end{aligned}$$

Thus there exists a positive number M such that if n is a positive integer, then

$$\sum_{p=k}^{k+n}|g_{2p+1}-g_{2p-1}| \geqq M^2\sum_{p=k}^{k+n}|b_{2p+1}|\,|D_{2p}|^2.$$

Hence the series $\sum|b_{2p+1}|\,|D_{2p}|^2$ converges. Thus since

$$\left|1-\frac{D_{2p+1}}{D_{2p-1}}\right| = \frac{|b_{2p+1}|\,|D_{2p}|^2}{|D_{2p}D_{2p-1}|},\qquad p\geqq k,$$

we see that the series $\sum_{(p\geqq k)}|1-D_{2p+1}/D_{2p-1}|$ converges. Hence $\{D_{2p-1}\}$ converges absolutely and $\lim D_{2p-1}\neq 0$. Now we suppose that the series $\sum|b_{2p-1}|$ diverges. Since the series $\sum|b_{2p+1}|\,|D_{2p}|^2$ converges, it follows that $\{D_{2p}\}$ contains an infinite subsequence which converges to zero. Thus if L is a positive number and N is a positive integer, there exists a positive integer q greater than N such that

$$|g_{2q+1}-g_{2q}| = \frac{1}{|D_{2q+1}D_{2q}|} > L.$$

But this contradicts the assumption that each of the sequences $\{g_{2p-1}\}$ and $\{g_{2p}\}$ converges. Hence (1.1) converges in case $\sum|b_{2p-1}|$ diverges. We next suppose that $\limsup|b_2+b_4+\cdots+b_{2p}|=\infty$. We see from (2.1) that if $n>k$, then

$$b_{2k}+b_{2k+2}+\cdots+b_{2n} = D_{2n}/D_{2n-1}-D_{2k-2}/D_{2k-1} + \sum_{p=k}^{n-1}\frac{b_{2p+1}D_{2p}^2}{D_{2p+1}D_{2p-1}}.$$

From this we conclude that $\{D_{2p}\}$ contains an unbounded subsequence since the series $\sum |b_{2p+1}| |D_{2p}|^2$ converges and $\{D_{2p-1}\}$ converges to a point not zero. Hence if $e>0$ and N is a positive integer, there exists a positive integer i greater than N such that

$$|g_{2i+1} - g_{2i}| = \frac{1}{|D_{2i+1}D_{2i}|} < e.$$

But this contradicts our assumption that each of the sequences $\{g_{2p-1}\}$ and $\{g_{2p}\}$ converges but (1.1) does not converge. Thus the theorem is established for the case we considered. In case we assume that $\{g_{2p}\}$ converges absolutely and $\{g_{2p-1}\}$ converges, we can show in an analogous way that (1.1) converges if $\sum |b_{2p}|$ diverges or if $\limsup |b_1+b_3+ \cdots +b_{2p-1}| = \infty$. Thus the proof of Theorem A is complete.

THEOREM B. *If the series $\sum |b_{2p+1}| |D_{2p}|^2$ converges and there exists a positive integer k such that $\{g_p\}_{p=k}^{\infty}$ is a bounded complex number sequence, then neither* (1) *nor* (2) *of condition* (H) *can hold and* (1.1) *converges if, and only if, b satisfies* (3) *of condition* (H).

PROOF. Let t be a positive integer such that $D_{2j}\neq 0$, $D_{2j-1}\neq 0$ if $j\geqq t$. As in the proof of Theorem A, the series $\sum |1-D_{2p+1}/D_{2p-1}|$ converges. Thus $\{D_{2p-1}\}$ converges absolutely and $\lim D_{2p-1}\neq 0$. As in the proof of Theorem A if we assume that the series $\sum |b_{2p-1}|$ diverges then we contradict the assumption that there exists a positive integer k such that $\{g_p\}_{p=k}^{\infty}$ is a bounded complex number sequence. Thus (1) of condition (H) cannot hold. Therefore, the series $\sum |b_{2p-1}| |D_{2p}|$ converges. By (2.1), if $p>t$, then

$$\begin{aligned} & b_{2t} + b_{2t+2} + \cdots + b_{2p} \\ (2.3) \qquad & = D_{2p}/D_{2p-1} - D_{2t-2}/D_{2t-1} + \sum_{q=t}^{p-1} \frac{b_{2q+1}D_{2q}^2}{D_{2q+1}D_{2q-1}}. \end{aligned}$$

Let R be a number such that if $i>t$, then

$$\left| \sum_{p=t}^{i-1} \frac{b_{2p+1}D_{2p}^2}{D_{2p+1}D_{2p-1}} - D_{2t-2}/D_{2t-1} \right| < R.$$

Then

$$\begin{aligned} & |b_{2p+1}(b_{2t} + b_{2t+2} + \cdots + b_{2p})^2| \\ & \leqq \frac{1}{|D_{2p-1}|^2} |b_{2p+1}| |D_{2p}|^2 + \frac{2R}{|D_{2p-1}|} |b_{2p+1}D_{2p}| + R^2 |b_{2p+1}|. \end{aligned}$$

Hence we see that the series $\sum|b_{2p+1}(b_{2t}+b_{2t+2}+\cdots+b_{2p})^2|$ converges. Therefore, (2) of condition (H) cannot hold. Thus we see that by the result of Scott and Wall mentioned in the proof of Theorem A, if (1.1) converges, then (3) of condition (H) holds. We now assume that (3) of condition (H) holds. Then by (2.3), $\lim|D_{2p}|=\infty$, and since $\lim D_{2p-1}\neq 0$, we see that if e is a positive number, there exists a number N such that if $n>N$, then

$$|g_{2n+1}-g_{2n}| = \frac{1}{|D_{2n+1}D_{2n}|} < e.$$

But by (2.2), $\{g_{2p-1}\}$ converges absolutely. Hence (1.1) converges. This completes the proof of the theorem.

We now use Theorem B in order to prove the theorem of Scott and Wall mentioned in the introduction.

THEOREM C (SCOTT AND WALL). *Let $k_1, k_2, k_3, \cdots$ be constants such that $k_1>0$, $k_{2p+1}\geqq 0$, $\mathrm{Re}(k_{2p})\geqq 0$, $p=1, 2, 3, \cdots$, and let $z_1, z_2, z_3, \cdots$ be complex variables. The continued fraction* (1.3) *converges for* $\mathrm{Re}\,(z_p)\geqq d$, $|z_p|<M$, $p=1, 2, 3, \cdots$, *where d and M are any positive constants, if, and only if, the sequence $k_1, k_2, k_3, \cdots$ satisfies condition* (H).

PROOF. We observe that $k_1z_1, k_2, k_3z_2, \cdots$ satisfies condition (H) if, and only if, $\{k_p\}$ satisfies condition (H). Thus we need only consider the proof of the sufficiency. Also we see that the series $\sum k_{2p+1}\,\mathrm{Re}\,z_{p+1}|D_{2p}|^2$ and $\sum k_{2p+1}|z_{p+1}|\,|D_{2p}|^2$ converge or diverge together, where D_q is the denominator of the qth approximant of (1.3). For simplicity of notation, we let $b_{2p+1}=k_{2p+1}z_{p+1}$ and $b_{2p+2}=k_{2p+2}$, $p=0, 1, 2, \cdots$. Let $t_p(u)\ 1/(b_p+u)$, $p=1, 2, 3, \cdots$. If p is a positive integer and $\mathrm{Re}\,u\geqq 0$, then $\mathrm{Re}\,t_p(u)\geqq 0$. Let H denote the half-plane $z+\bar{z}\geqq 0$. If n is a positive integer, let $T_n(u)=t_1t_2\cdots t_n(u)$. Then $T_n(H)$ is a circular disc since $b_1\neq 0$ and we denote its radius by R_n. Then $T_{p+1}(H)$ is a subset of $T_p(H)$, $p=1, 2, 3, \cdots$, and

$$T_n(u) = \frac{C_{n-1}u+C_n}{D_{n-1}u+D_n},$$

$$R_n = \frac{1/2}{\mathrm{Re}\,\overline{D}_nD_{n-1}} = \frac{1/2}{\sum_{p=1}^{n}\mathrm{Re}\,b_p\,|D_{p-1}|^2}.$$

Thus $T_n(0)=C_n/D_n=g_n$ and $T_n(\infty)=C_{n-1}/D_{n-1}=g_{n-1}$, where g_q is the qth approximant of (1.3). We now examine R_{2n},

$$R_{2n} = \frac{1/2}{\sum_{p=0}^{n-1} \operatorname{Re} b_{2p+1} |D_{2p}|^2 + \sum_{p=1}^{n} \operatorname{Re} b_{2p} |D_{2p-1}|^2}$$

$$= \frac{1/2}{\sum_{p=1}^{n-1} k_{2p+1} \operatorname{Re} z_{p+1} |D_{2p}|^2 + \sum_{p=1}^{n} \operatorname{Re} b_{2p} |D_{2p-1}|^2}.$$

If (1) or (2) of condition (H) holds, then by Theorem B, the series $\sum k_{2p+1}|z_{p+1}|\,|D_{2p}|^2$ diverges, and so the series $\sum k_{2p+1} \operatorname{Re} z_{p+1}|D_{2p}|^2$ diverges. Thus $R_{2n} \to 0$ as $n \to \infty$ and (1.3) converges. Suppose the series $\sum k_{2p+1} \operatorname{Re} z_{p+1}|D_{2p}|^2$ converges. Then the series $\sum k_{2p+1}|z_{p+1}| \cdot |D_{2p}|^2$ converges, and so by Theorem B, (1.3) converges if $k_1 z_1, k_2, k_3 z_2, \cdots$ satisfies (3) of condition (H). Thus the proof of Theorem C is complete.

References

1. W. T. Scott and H. S. Wall, *On the convergence and divergence of continued fractions*, Amer. J. Math. vol. 69 (1947) pp. 551–561.

2. H. S. Wall, *Analytic theory of continued fractions*, New York, 1948.

3. ———, *Some convergence problems for continued fractions*, Amer. Math. Monthly, vol. 64 (1957) pp. 95–103.

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