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# ON A DIVERGENT TRIGONOMETRICAL SERIES GIVEN BY STEINHAUS[1]

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Steinhaus [**1**; **2**, p. 283] gave the series

$$\sum_{n=2}^{\infty} (\log n)^{-1} \cos n(x - \log \log n) \tag{1}$$

as an example of an everywhere-divergent trigonometric series with coefficients tending to zero. Plainly, a *sine* series cannot diverge everywhere, since it must converge whenever $x \equiv 0 (\text{mod } \pi)$. There is, however, no *a priori* reason why a *cosine* series should not diverge everywhere. It is not immediately clear from Steinhaus's argument [**1**] whether the "cosine part" of (1), namely

$$\sum_{n=2}^{\infty} (\log n)^{-1} \cos (n \log \log n) \cos nx, \tag{2}$$

has any points of convergence. Accordingly I exhibit here a class of everywhere-divergent cosine series, of which (2) is a special case.

Theorem. *Suppose that $u(n) \uparrow \infty$, $c_n \downarrow 0$ as $n \to \infty$, and that there exists a sequence of positive integers $\{p_n\}$ such that*

$$\limsup_{n\to\infty} (n + p_n)\{u(n + p_n) - u(n)\} < \frac{1}{2}, \tag{3}$$

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$$\liminf_{n\to\infty} \sum_{n+1}^{n+p_n} c_r > 0. \tag{4}$$

*Then the cosine series*

$$\sum_{n=1}^{\infty} c_n \cos nu(n) \cos nx \tag{5}$$

*diverges for all real $x$, and the sine series*

$$\sum_{n=1}^{\infty} c_n \sin nu(n) \sin nx \tag{6}$$

*diverges for all* $x \not\equiv 0 \pmod{\pi}$.

The method used to prove this theorem is a refinement of that employed by Steinhaus to prove the divergence of (1). Put

$$A_r(x) = \cos ru(r) \cos rx - \frac{1}{2}\cos 2rx \tag{7}$$

and suppose that

$$u(n) \leqq x < u(n+1), \qquad n+1 \leqq r \leqq n+p_n.$$

Then

$$\begin{aligned}\left| A_r(x) - \frac{1}{2} \right| &= \left| \cos rx\{\cos ru(r) - \cos rx\} \right| \\ &= \left| 2\cos rx \sin \frac{1}{2} r\{x+u(r)\} \sin \frac{1}{2} r\{x-u(r)\} \right| \\ &\leqq r \left| x - u(r) \right| \\ &\leqq (n+p_n)\{u(n+p_n) - u(n)\}.\end{aligned}$$

By (3) there exist an integer $n_0$ and a number $\lambda < 1/2$ such that this is less than $\lambda$ for all $n > n_0$. Thus for all $n$ and $x$ satisfying

$$n > n_0, \qquad u(n) \leqq x < u(n+1), \tag{8}$$

we have

$$\sum_{n+1}^{n+p_n} c_r A_r(x) > \left(\frac{1}{2} - \lambda\right) \sum_{n+1}^{n+p_n} c_r. \tag{9}$$

Every value of $x$ satisfies (8), modulo $2\pi$, for an infinity of values of $n$, and the left-hand side of (9) has period $2\pi$. Thus, for every $x$, (9) is true for an infinity of $n$. Hence by (4),

$$\limsup_{n\to\infty} \sum_{n+1}^{n+p_n} c_r A_r(x) > 0.$$

But since $c_n \downarrow 0$, we have for every $x$

$$\liminf_{n\to\infty} \sum_{n+1}^{n+p_n} c_r \cos 2rx \geqq 0,$$

and so by (7),

$$\limsup_{n\to\infty} \sum_{n+1}^{n+p_n} c_r \cos ru(r) \cos rx > 0.$$

By the general principle of convergence, this proves the divergence of (5).

I omit the proof of the divergence of (6) for all $x \not\equiv 0 \pmod{\pi}$, which is similar. This proves the theorem.

To deduce from the theorem the divergence of (2) everywhere, we take

$$u(n) = \log\log n, \qquad c_n = (\log n)^{-1} (n \geqq 2), \qquad p_n = \left[\frac{1}{4}\log n\right],$$

where the square brackets denote the integer part. Then $u(n) \uparrow \infty$, $c_n \downarrow 0$, and the left-hand sides of (3) and (4) are both 1/4. Thus (2) diverges everywhere, by the theorem. Similarly the "sine part" of (1) diverges for all $x \not\equiv 0 \pmod{\pi}$.

It is not worth while to examine in detail the order of magnitude of those sequences $\{c_n\}$ for which $u(n)$ and $p_n$ can be chosen to satisfy (3) and (4). However, it is easy to show that the choice

$$c_n = (\log n)^{\alpha_1} (\log_2 n)^{\alpha_2} (\log_3 n)^{\alpha_3} \cdots (\log_q n)^{\alpha_q},$$

where $q$ is a fixed integer and $\log_m$ denotes the $m$th iterated logarithm, is permissible if $c_n \downarrow 0$ and $\sum n^{-1} c_n = \infty$, but not otherwise.

*Note added in proof*, December 4, 1958. If, in (5), we replace $\cos nu(n)$ by $2+\cos nu(n)$, we obtain, as may easily be seen, an everywhere-divergent cosine series with *non-negative* coefficients tending to zero.

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