A GEOMETRIC CONSTRUCTION OF THE M-SPACE CONJUGATE TO AN L-SPACE[1]

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1. **Introduction.** In a previous paper [**3**] the author described a geometric construction of (abstract) L-spaces based upon the characterization of such spaces by R. E. Fullerton [**6**]. The construction to be introduced here is analogously motivated by the geometric characterization of C-spaces given by J. A. Clarkson [**4**]. Clarkson showed that the unit sphere in a C-space is the intersection of a translate of the positive cone with the negative of that translate. Letting C denote the positive cone of an L-space X with F-unit e we will show that the space of all points lying on rays from the origin Θ through the set $S=(C-e)\cap(e-C)$, when given an appropriate norm and partial ordering, is an M-space with unit element e and unit sphere S and is lattice isomorphic and isometric to the space $L^\infty(\Omega, m)$ conjugate to the concrete representation $L(\Omega, m)$ of X. To do this we rely heavily upon the results in the classic papers of S. Kakutani [**7, 8**].

2. **Preliminaries.** Let X be a real Banach space with norm $\|x\|$ and additive identity element Θ which is a linear lattice under the partial ordering relation $x\geqq y$ (cf. G. Birkhoff [**2**]). Let $x\vee y$ and $x\wedge y$ denote the lattice least upper bound and greatest lower bound, respectively, of the elements x, $y\in X$. S. Kakutani [**8**] calls such a space an (abstract) *M-space* if, in addition, the following conditions are satisfied:

(K.7) $\quad x_n \geqq y_n,\ \|x_n - x\| \to 0,\ \|y_n - y\| \to 0$ imply $x \geqq y$.

(K.8) $\quad x \wedge y = \Theta$ implies $\|x + y\| = \|x - y\|$.

(K.9) $\quad x \geqq \Theta,\ y \geqq \Theta$ imply $\|x \vee y\| = \max(\|x\|, \|y\|)$.

If condition (K.9) is replaced by

(K.10) $\quad x \geqq \Theta,\ y \geqq \Theta$ imply $\|x + y\| = \|x\| + \|y\|$,

Kakutani [**7**] calls the space an (abstract) *L-space.*

If we let $C=\{x: x\geqq\Theta\}$ denote the positive cone in X, condition

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(K.7) above may be restated as: "C is closed in the norm topology."[2]

An element e of a linear lattice X is called an *F-unit* if $e>\Theta$ and if $e\wedge x>\Theta$ for any $x>\Theta$. A *unit element* of a Banach lattice X is an element $e\geqq\Theta$ such that $\|e\|=1$ and $x\leqq e$ for any $x\in X$ with $\|x\|\leqq 1$. (Cf. [8, p. 997]). It should be recalled from elementary linear space theory that if a convex sub set S of a linear space X is radial at Θ, symmetric about Θ, and circled then the *Minkowski functional* $p(x) =\inf \{a: x/a\in S, a>0\}$ is a pseudo-norm on X. (Cf. [5]).

Let $L(\Omega, m)$ denote the space of all measurable real valued functions $x(t)$ which are integrable over a compact Hausdorff space Ω with respect to a completely additive measure m such that $m(\Omega)=1$. A well known result of S. Kakutani [7] tells us that for any L-space X with F-unit there exists such a space $L(\Omega, m)$ lattice isomorphic and isometric to X and called the *concrete representation* of X. We notice that if e is an F-unit of X and a is any positive real number then ae is also an F-unit. Hence we shall assume without loss of generality that the F-unit distinguished in X has unit norm.

Since, in the sequel, certain results we wish to obtain concerning subsets of the L-space X will be more easily demonstrated by using their images in $L(\Omega, m)$ and since no confusion nor loss of generality will occur, we will often replace the subset of X by its image in the concrete representation of X without changing notation and without specifically mentioning that the implied equality is in actuality a lattice isomorphism and isometry.

3. The construction.

THEOREM. *Let X be an L-space with F-unit e, partial ordering $x\geqq y$, positive cone C, norm $\|x\|$ and unit sphere $U=\{x: \|x\|\leqq 1, x\in X\}$. Define $S=(C-e)\cap(e-C)$, $p(x)=\inf\{a: x/a\in S, a>0\}$ and $Y =\{aS: a\geqq 0\}$. Then Y is an M-space with unit element e, partial ordering $x\geqq y$, positive cone $C\cap Y$, norm $p(x)$ and unit sphere $S =\{x: p(x)\leqq 1, x\in Y\}\subset U$. Furthermore, Y is lattice isomorphic and isometric to the space $L^\infty(\Omega, m)$ conjugate to the concrete representation $L(\Omega, m)$ of X.*

In order to clarify what would otherwise be a lengthy proof we will demonstrate the theorem by proving the following sequence of lemmas.

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[2] It has been communicated to the author by J. E. Kist that axiom (K.7) is redundant since the positive cone C in any real Banach lattice is strongly closed. To see this we need only note that the positive cone is the set $C=\{x: x\in X, x-|x|=\Theta\}$ and hence is the inverse image of the closed set $\{\Theta\}$ under the continuous map $x\rightarrow x-|x|$.

3.1. LEMMA. *Y is a normed linear space with norm $p(x)$ and unit sphere $S\subset U$.*

PROOF. Using the obvious convexity and symmetry of S it is easy to verify that Y is a real linear space. The fact that S is also circled and radial at Θ in Y shows that $p(x)$ is a pseudo-norm on Y. Let x be any element of S. Then, since $S=(C-e)\cap(e-C)$, we have $x=ay-e=e-bz$ where y and z are elements in C of unit norm and a and b are nonnegative real numbers. Now $2x=ay-bz$ and $2e=ay+bz$ so $2\|x\|\leqq a\|y\|+b\|z\|=a+b$ and $2=2\|e\|=a\|y\|+b\|z\|=a+b$. (Here we use the facts that X is an L-space, in particular satisfying (K.10), and that $\|e\|=1$.) Thus $x\in S$ implies $\|x\|\leqq 1$, i.e. $S\subset U$. If $p(x)=0$ then $p(ax)=0$ for any real number a. It is an easily demonstated property of $p(x)$ that $\{x: p(x)<1\}\subset S$. Thus $p(x)=0$ implies $ax\in S\subset U$ for every real number a. This is possible only if $x=\Theta$ hence $p(x)$ is a norm on Y. To show that $S=\{x: x\in Y, p(x)\leqq 1\}$ we need only show that $p(x)=1$ implies $x\in S$. Suppose, therefore, that $x\in Y$ with $p(x)=1$. Define $y_n=(1-1/n)x$ for $n=1, 2, 3, \cdots$. Then $p(y_n)<1$ and $y_n\in\overline{S}$ (the closure of S relative to the norm topology of $\overline{X}$). Both $C-e$ and $e-C$ are closed sets in X and hence $S=\overline{S}$. Thus $x\in S$ and this lemma is proved.

3.2. LEMMA. *e is a unit element of Y.*

PROOF. The set $C\cap Y$ is clearly a cone in Y. We define a partial ordering in Y by: $x\geqq y$ if and only if $x-y\in C\cap Y$ and notice immediately that this is exactly the ordering in X restricted to Y. Certainly $e\geqq\Theta$ in Y since $e>\Theta$ in X. Since $e\in S$, by Lemma 3.1, $p(e)\leqq 1$. Suppose that $p(e)<1$. Then, by definition of $p(x)$, there exists a real number a, $0<a<1$, such that $1/a\cdot e\in S=(C-e)\cap(e-C)$. But $1/a\cdot e\in e-C$ implies $(a-1)e\in C$ and since $a-1<0$ this contradicts that $e\in C$. Thus $p(e)=1$. By definition of S, $x\in S$ if and only if $-e\leqq x\leqq e$ so $p(x)\leqq 1$ implies $x\leqq e$. This lemma is proved.

3.3. LEMMA. *In $L(\Omega, m)$ we have $S=\{x(t): |x(t)|\leqq 1$ a.e.$\}$, $p(x)=\inf\{a: |x(t)|\leqq a$ a.e., $a>0\}$, and $Y=\{x(t)$: there exists an $a>0$ for which $|x(t)|\leqq a$ a.e.$\}$.*

PROOF. As mentioned in §2. We make no distinction between sets in X and their equivalent images in $L(\Omega, m)$. Thus, saying that in $L(\Omega, m)$, $S=\{x(t): |x(t)|\leqq 1$ a.e.$\}$, we mean that the image of S under the equivalence mapping is exactly the set of all functions $x(t)\in L(\Omega, m)$ which are bounded in absolute value by 1 almost everywhere on Ω. That this is indeed the case follows from the facts that e corresponds to the function $e(t)\equiv 1$ a.e. (cf. [**7**]), $x\geqq\Theta$ in $L(\Omega, m)$

if and only if $x(t) \geqq 0$ a.e. and that $x \in S$ if and only if $-e \leqq x \leqq e$. The other two equations of the lemma follow easily.

3.4. LEMMA. *Y is complete under $p(x)$.*

PROOF. We identify Y with its image in $L(\Omega, m)$ and use Lemma 3.3. (Completeness is a property invariant under the equivalence mapping from X onto $L(\Omega, m)$.) If x is any element of Y then there exists an $a>0$ such that $|x(t)|<a$ a.e. and hence $x(t)$ is an essentially bounded function—i.e. $x(t)$ is in $L^\infty(\Omega, m)$. Let $\{x_n\}$ be any Cauchy sequence in Y. That is, given $\epsilon>0$ there exists an integer $N(\epsilon)$ such that $\inf\{a: |x_n(t)-x_m(t)|<a \text{ a.e., } a>0\}<\epsilon$ for all $n, m \geqq N(\epsilon)$. In particular, for $n, m \geqq N(\epsilon)$, $|x_n(t)-x_m(t)|<\epsilon$ a.e. so $\{x_n(t)\}$ is a Cauchy sequence in $L^\infty(\Omega, m)$. Since $L^\infty(\Omega, m)$ is complete in the norm $\|x\|_\infty = \text{essup}\,|x(t)| = \inf\{(\sup[|x(t)| : t \notin E]) : m(E)=0\}$ there exists a function $y(t) \in L^\infty(\Omega, m)$ such that given any $\epsilon>0$ there exists an $N'(\epsilon)$ such that $\|x_n - y\|_\infty < \epsilon$ when $n \geqq N'(\epsilon)$. Thus, for $n \geqq N'(\epsilon)$ we have $|x_n(t)-y(t)|<\epsilon$ a.e. and, by Lemma 3.3, $x_n - y \in Y$ so, clearly, $y \in Y$. Finally, since $|x_n(t)-y(t)|<\epsilon$ a.e. implies $p(x_n - z)<\epsilon$ we have that Y is complete. This lemma is proved.

3.5. LEMMA. *Y is an M-space.*

PROOF. That Y is a partially ordered linear space under the relation $x \geqq y$ defined by $x \geqq y$ if and only if $x-y \in C \cap Y$ is easily seen to be a consequence of the fact that $C \cap Y$ is a cone such that $[C \cap Y] \cap [-(C \cap Y)] = \{\Theta\}$. To show that $x \vee y$ exists in Y consider again the concrete representation $L(\Omega, m)$ of X. In $L(\Omega, m)$, $(x \vee y)(t) = \max[x(t), y(t)]$. If x and y are in Y then there exists real numbers $a>0$ and $b>0$ such that $|x(t)| \leqq a$ a.e. and $|y(t)| \leqq b$ a.e. Then $|(x \vee y)(t)| \leqq \max(a, b)$ a.e. so $x \vee y \in Y$. Clearly $x \wedge y = -[(-x) \vee (-y)]$. Thus Y is a linear lattice. We shall verify conditions (K.8) and (K.9) to complete the proof of this lemma. To prove (K.8) and (K.9) are satisfied we first use a technique of Clarkson [**4**]. Assert that for any $k>0$, $p(x) \leqq k$ if and only if $-ke \leqq x \leqq ke$. Secondly assert that for any $x \in Y$, $-p(x)e \leqq x \leqq p(x)e$. To prove the first assertion we note that $p(x) \leqq k$ if and only if $x/k \in S$ or, equivalently, $-e \leqq x/k \leqq e$. The second assertion is trivial for $x=\Theta$ and if $x \neq \Theta$ it follows from the first by setting $k=p(x)$. Now assume that $\Theta \leqq x \leqq y$ in Y. If $y=\Theta$ certainly $p(x) \leqq p(y)$. If $y \neq \Theta$ then $-p(y)e \leqq x \leqq y \leqq p(y)e$ and, by the first assertion above, $p(x) \leqq p(y)$. We now prove that (K.9) is satisfied. Assume that $x \geqq \Theta$, $y \geqq \Theta$. Since $x \leqq x \vee y$ and $y \leqq x \vee y$ we have $p(x) \leqq p(x \vee y)$ and $p(y) \leqq p(x \vee y)$, and hence $\max[p(x), p(y)] \leqq p(x \vee y)$. Without loss of generality assume that

$0 \leqq p(y) \leqq p(x)$. Since $x \leqq p(x)e$ and $y \leqq p(y)e \leqq p(x)e$ we have $x \vee y \leqq p(x)e$ so that $-p(x)e \leqq x \vee y \leqq p(x)e$ and $p(x \vee y) \leqq p(x) = \max [p(x), p(y)]$. Finally, to prove (K.8) let $x \wedge y = \Theta$ and again assume that $0 \leqq p(y) \leqq p(x)$. Now $x+y = x \vee y$ so $p(x+y) = p(x \vee y) = \max [p(x), p(y)] = p(x)$. Also $2x = (x+y)+(x-y)$ so $2p(x) \leqq p(x+y)+p(x-y) = p(x)+p(x-y)$. Thus $p(x) = p(x+y) \leqq p(x-y)$. To get the reverse inequality note that $-p(x+y)e \leqq -(x+y) \leqq -y \leqq x-y \leqq x \leqq x+y \leqq p(x+y)e$ so $p(x-y) \leqq p(x+y)$. Thus (K.8) holds and this lemma is proved.

3.6. LEMMA. *Y is lattice isomorphic and isometric to the space $L^\infty(\Omega, m)$ conjugate to the concrete representation $L(\Omega, m)$ of X.*

PROOF. We saw in Lemma 3.3 that the image in $L(\Omega, m)$ of Y under the equivalence mapping between X and $L(\Omega, m)$ is exactly the set $\{x(t): x(t) \in L(\Omega, m), |x(t)| \leqq a \text{ a.e. for some } a > 0\}$. But this is precisely the set of essentially bounded m-measurable functions on Ω. That the isomorphism between Y and $L^\infty(\Omega, m)$ is order preserving is clear since the ordering in $L^\infty(\Omega, m)$ is the restriction of the ordering in $L(\Omega, m)$. Finally that the mapping is an isometry is seen by noticing that $\{x(t): |x(t)| \leqq 1 \text{ a.e.}\}$ coincides with the unit sphere in $L^\infty(\Omega, m)$. This lemma and the theorem are proved.

3.7. COROLLARY. *For any two choices e_1 and e_2 of the F-unit in X the corresponding M-spaces Y_1 and Y_2 are isometric. Furthermore, the corresponding measure spaces are in one-to-one measure preserving correspondence modulo sets of measure zero.*[3]

4. REMARK. If the L-space X contains no F-unit then X is a direct sum $\sum_\alpha \{X_\alpha\}$ of L-spaces X_α each with an F-unit (cf. S. Kakutani [7]). We apply the construction of the theorem to each coordinate space X_α to arrive at its conjugate space X_α^*. To get X^*, the conjugate of X, we take the direct product $\prod_\alpha \{X_\alpha^*\}$ of the constructed conjugates.

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[3] The referee helpfully suggested that this corollary be added.

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PURDUE UNIVERSITY AND
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ON $u''+(1+\lambda g(x))u=0$ FOR $\int_0^\infty |g(x)|\,dx<\infty$

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1. Bellman [1] has raised several questions concerning the solution $u(x, \lambda)$ of

$$u'' + [1 + \lambda g(x)]u = 0, \qquad u(0) = 0, \qquad u'(0) = 1 \tag{1.0}$$

when

$$\int_0^\infty |g(x)|\,dx < \infty. \tag{1.1}$$

He states that it is known that for real λ

$$\lim_{x\to\infty} \{u(x, \lambda) - r(\lambda) \sin [x + \theta(\lambda)]\} = 0$$

where r and θ are functions of λ. He asks for the analytic properties of r and θ if λ is a complex variable. In particular he asks whether, if $g>0$, the nearest singularity of r or θ to the origin $\lambda=0$, is on the negative real axis. It will be shown below that it is not. Indeed if g is real, r and θ are analytic functions of λ for real λ.

Let $g(x)$ be piecewise continuous, (Lebesgue integrable would suffice), and satisfy (1.1). Let

$$B(x) = \int_0^x |g(\xi)|\,d\xi. \tag{1.2}$$

THEOREM. *There is a solution $u(x, \lambda)$ of* (1.0) *which for each x is an entire function of λ and which satisfies*

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