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ON GREEN'S THEOREM

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Green's theorem in two dimensons says that if C is a simple closed curve bounding the region Q, if $A(x, y)$ and $B(c, y)$ are continuous functions having derivatives, then under suitable further conditions we have,

$$\int_C A\,dx + B\,dy = \int\int_Q \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dxdy \tag{1}$$

where the line integral is taken in a positive sense around the curve C. In [1] Bochner investigated under which conditions (1) holds. There it was shown that if A and B have certain regularity properties and if the integrand on the right of (1) behaves well, then (1) does hold. Here we shall prove (1) under what may be regarded as the weakest possible hypotheses. This question was also treated by Shapiro in [3], and though he assumes certain regularity of A and B, namely, the existence of the differential, he allows certain exceptional sets which we cannot allow. The proof of our theorem is modeled after the proof of the Looman-Mensov theorem as contained, for example, in [2]. We will not deal with the topological difficulties involved so that our theorem will only treat the case in which Q is a rectangle, whose sides are parallel to the coordinate axes.

THEOREM. *Let $A(x, y)$ and $B(x, y)$ be two functions defined on the rectangle Q, and continuous on the closure of Q. Assume further that the partial derivatives*

$$\frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}, \frac{\partial B}{\partial x}, \frac{\partial B}{\partial y}$$

exist everywhere in the interior of Q, except perhaps at a countable num-

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ber of points. If $\partial B/\partial x - \partial A/\partial y$ *is Lebesgue integrable in the rectangle* Q, *then* (1) *holds.*

PROOF. We first need a lemma which is contained in [2]. In the following, the word "rectangle" means the direct product of two intervals. Here we use the notation $|S|$ to represent the measure of the set S.

LEMMA. *Let* $w(x, y)$ *be a function defined in a square* Q, *such that* $\partial w/\partial x$ *and* $\partial w/\partial y$ *exist at all but an enumerable set in* Q. *Let* F *be a closed, nonempty set in* Q, *and* N *a finite constant such that*

$$\begin{aligned} |w(x, y + k) - w(x, y)| &\leqq N|k|, \\ |w(x + h, y) - w(x, y)| &\leqq N\cdot|h| \end{aligned} \tag{2}$$

whenever (x, y) *belong to* F *and* $(x+h, y)$ *and* $(x, y+k)$ *belong to* Q. *Let* J *be the smallest rectangle containing* F *and assume* J *is the product of* (a_1, b_1) *and* (a_2, b_2). *Then*

$$\begin{aligned} \left| \iint_F \frac{\partial w}{\partial x} dxdy - \int_{a_2}^{b_2} [w(b_1, y) - w(a_1, y)]dy \right| &\leqq 5N\cdot|Q - F|, \\ \left| \iint_F \frac{\partial w}{\partial y} dxdy - \int_{a_1}^{b_1} [w(x, b_2) - w(x, a_2)]dx \right| &\leqq 5N\cdot|Q - F|. \end{aligned} \tag{3}$$

It is clearly enough to prove the theorem for all rectangles Q' properly contained in Q, for, in this case, we can approximate Q from the interior by a sequence of such rectangles, and for each of which (1) holds. Now by the Lebesgue integrability of $\partial B/\partial x - \partial A/\partial y$ the right side approaches the corresponding integral over Q, and by the continuity of A and B so does the left side. Hence we may assume in the original statement of the theorem that A and B are actually defined in a neighborhood of Q where they are continuous and have derivatives at all but a countable number of points. Q will now denote the closed rectangle. Now, let E be the set of all points P in Q, such that (1) holds for integrations taken over all rectangles in a sufficiently small neighborhood of P. We shall show that E is all of Q. Let $F = Q - E$. Since E is obviously open, F is closed. Let H_n be the set of all points (x, y) in Q, such that

$$\left|\frac{A(x + h, y) - A(x, y)}{h}\right|, \quad \left|\frac{A(x, y + k) - A(x, y)}{k}\right|,$$

$$\left|\frac{B(x + h, y) - B(x, y)}{h}\right|, \quad \left|\frac{B(x, y + k) - B(x, y)}{k}\right|$$

are all bounded by n whenever $|h| \leqq 1/n$, $|k| \leqq 1/n$, and all the quantities involved are defined. Clearly Q, with a countable number of exceptions, is the union of all these closed sets H_n. We may assume that F has points in the interior of Q, since otherwise we may approximate Q by a sequence of interior squares, as described above. Therefore by the Baire category theorem, since F is also closed, either F contains an isolated point in the interior of Q, or there is some square I of edge smaller than $1/N$, in the interior of Q, such that $I \cap F$ is nonempty and is contained in H_N for some N. If a rectangle lies completely in E, then the Heine-Borel theorem shows that (1) holds for it. Hence we see that isolated points of F cannot occur and so the second alternative holds. Then the conditions of the lemma hold and we have

$$\left| \int_{\partial J} A dx + B dy - \iint_{J \cap F} \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx dy \right| \leqq 10N \cdot |I - F|. \tag{4}$$

Here J is the smallest rectangle containing $I \cap F$ and ∂J denotes the boundary of J. The set $I - J$ is a finite union of rectangles each of which can be approximated from the interior by rectangles wholly contained in E. Hence (1) holds for the set $I - J$, where the line integral is taken around its boundary in the positive sense. Thus we have

$$\int_{\partial(I-J)} A dx + B dy = \iint_{I-J} \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx dy. \tag{5}$$

So by (4) we have that

$$\left| \int_{\partial I} A dx + B dy - \iint_{(J \cap F) \cup (I-J)} \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx dy \right| \leqq 10N \cdot |I - F|. \tag{6}$$

From (6) it follows that

$$\left| \int_{\partial I} A dx + B dy \right| \leqq 10N \cdot |I| + \iint_I \left| \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right| dx dy. \tag{7}$$

Now (7) holds equally well for any square I' contained in I. Thus the set function which assigns to every square I' the quantity

$$\int_{\partial I'} A dx + B dy$$

is dominated by an absolutely continuous measure and hence extends to an absolutely continuous measure defined on all Borel sets.

Thus, it is given by the indefinite integral of some function. If we can then show that the derivative of this measure in the sense of averages taken over smaller and smaller squares is equal to $\partial B/\partial x - \partial A/\partial y$ almost everywhere, then we will know that (1) holds for all rectangles in I and hence that $I \cap F$ is empty, which is a contradiction. Now the derivative of the measure at almost all points not in F is clearly $\partial B/\partial x - \partial A/\partial y$. This is merely the theorem concerning differentiation of indefinite integrals, since at such points (1) does hold. On the other hand, if P is a point of density of F, then for sufficiently small squares I around P, we have that $|I-F|/|I|$ is arbitrarily small. Thus from (6) it follows that $(1/|I|)\int_{\partial I} A\,dx + B\,dy$ approaches $(1/|I|)\int\int_{(J\cap F)\cup(I-J)}(\partial B/\partial x - \partial A/\partial y)dxdy$. This last quantity approaches the derivative of the integral taken with respect to the sets $(J\cap F)\cup(I-J)$. These are a regular sequence of sets in the sense of [**2** p. 106], since $|(J\cap F)\cup(I-J)|/|I|$ tends to 1, and so this derivative is equal to $\partial B/\partial x - \partial A/\partial y$ at almost all points of F. Thus the measure is the desired indefinite integral and so (1) holds at all points P. Now by the Heine-Borel theorem it follows that (1) holds for the rectangle Q itself.

We may easily generalize this result to any number of dimensions.

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