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ON \( u'' + (1 + \lambda g(x)) u = 0 \) FOR \( \int_0^\infty |g(x)| \, dx < \infty \)

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1. Bellman [1] has raised several questions concerning the solution \( u(x, \lambda) \) of

\[
(1.0) \quad u'' + [1 + \lambda g(x)] u = 0, \quad u(0) = 0, \quad u'(0) = 1
\]

when

\[
(1.1) \quad \int_0^\infty |g(x)| \, dx < \infty.
\]

He states that it is known that for real \( \lambda \)

\[
\lim_{x \to \infty} \{ u(x, \lambda) - r(\lambda) \sin [x + \theta(\lambda)] \} = 0
\]

where \( r \) and \( \theta \) are functions of \( \lambda \). He asks for the analytic properties of \( r \) and \( \theta \) if \( \lambda \) is a complex variable. In particular he asks whether, if \( g > 0 \), the nearest singularity of \( r \) or \( \theta \) to the origin \( \lambda = 0 \), is on the negative real axis. It will be shown below that it is not. Indeed if \( g \) is real, \( r \) and \( \theta \) are analytic functions of \( \lambda \) for real \( \lambda \).

Let \( g(x) \) be piecewise continuous, (Lebesgue integrable would suffice), and satisfy (1.1). Let

\[
(1.2) \quad B(x) = \int_0^x |g(\xi)| \, d\xi.
\]

**Theorem.** There is a solution \( u(x, \lambda) \) of (1.0) which for each \( x \) is an entire function of \( \lambda \) and which satisfies

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ON $u'' + (1 + \lambda g(x))u = 0$ FOR $\int_0^\infty |g(x)| \, dx < \infty$

(1.3) $|u(x, \lambda)| \leq e^{\lambda |B(x)|}$.

Let

(1.4) $F(\lambda) = 1 - \lambda \int_0^\infty e^{-ix} g(x) u(x, \lambda) \, dx$,

(1.5) $G(\lambda) = 1 - \lambda \int_0^\infty e^{ix} g(x) u(x, \lambda) \, dx$.

Then $F$ and $G$ are entire functions of $\lambda$ and $F(0) = G(0) = 1$. Let

(1.6) $r(\lambda) = (F(\lambda)G(\lambda))^{1/2}$, $r(0) = 1$,

(1.7) $\theta(\lambda) = -\frac{1}{2i} \log \frac{F(\lambda)}{G(\lambda)}$, $\theta(0) = 0$.

Then if $\lambda$ is not a zero of $F$ or $G$

(1.8) $u(x, \lambda) - r(\lambda) \sin (x + \theta(\lambda)) \to 0$ as $x \to \infty$.

If $g(x)$ is real then $G(\lambda) = \overline{F(\lambda)}$ for all $\lambda$ and $F(\lambda) \neq 0$ for real $\lambda$. Hence the same is true for $G(\lambda)$ and therefore $r(\lambda)$ and $\theta(\lambda)$ are analytic in $\lambda$ for real $\lambda$ and are real for real $\lambda$. (Hence $r(\lambda)$ and $\theta(\lambda)$ have no singularities on the real axis if $g$ is real and $r(\lambda) > 0$ for real $\lambda$.) If $g(x) \geq 0$ and $\int_0^\infty g(x) \, dx > 0$ then the zeros of $G(\lambda)$ occur in the lower half-plane $\Re \lambda < 0$ (and those of $F(\lambda) = \overline{G(\lambda)}$ in the upper half-plane $\Re \lambda > 0$).

Since $F$ and $G$ are entire it follows from (1.6) and (1.7) that $r$ and $\theta$ have possible branch points at the singularities of $F$ and $G$ and otherwise are analytic. (The determination of $r$ and $\theta$ at a point $\lambda_0$ where $F$ and $G \neq 0$ is not unique but (1.8) will hold if both $r$ and $\theta$ are obtained by an analytic continuation of (1.6) and (1.7) from $\lambda = 0$ to $\lambda = \lambda_0$ along a path that avoids zeros of $F$ and $G$.)

Remark. From (1.3) and (1.4) and (1.5) it follows that nearest zero of $F$ or $G$ to $\lambda = 0$ is at least a distance

$$\frac{\log 2}{\int_0^\infty |g(x)| \, dx}$$

from $\lambda = 0$.

2. Lemma 1. The integral equation

(2.0) $u(x, \lambda) = \sin x - \lambda \int_0^x \sin (x - \xi) g(\xi) u(\xi, \lambda) \, d\xi$
has a solution which satisfies (1.0), is continuous in \((x, \lambda)\) for \(0 \leq x < \infty\) and \(|\lambda| < \infty\), and entire in \(\lambda\) for each \(x\). Also

\[ |u(x, \lambda)| \leq e^{B(\lambda)} \]

**Proof.** Let \(u_0(x, \lambda) = 0\) and use successive approximations.

**Lemma 2.** Let \(F\) and \(G\) be defined by (1.4) and (1.5). Then \(F\) and \(G\) are entire functions of \(\lambda\) and

\[ u(x, \lambda) = \frac{1}{2i} \left[ F(\lambda)e^{ix} - G(\lambda)e^{-ix} \right] + J \]

where

\[ \lim_{\lambda \to \infty} J(x, \lambda) = 0. \]

**Proof.** That \(F\) and \(G\) are entire follows from (1.1), (1.3) and (1.4) and (1.5). In (2.0) express \(\sin x\) and \(\sin (x - \xi)\) in exponential form and let

\[ J(x, \lambda) = \lambda \int_x^\infty \sin (x - \xi) g(\xi) u(\xi, \lambda) d\xi. \]

Then (2.1) follows from (1.4) and (1.5). From (1.1) and (1.3) follows

\[ |J(x, \lambda)| \leq e^{B(\lambda)} - e^{B(\xi)} \]

which implies (2.2).

**Lemma 3.** Let \(r(\lambda) = (F(\lambda)G(\lambda))^{1/2}, r(0) = 1, \theta(\lambda) = (1/2i) \log F(\lambda)/G(\lambda), \theta(0) = 0.\)

Then, if \(\lambda\) is not a zero of \(F\) or \(G\)

\[ \frac{1}{2i} \left[ F(\lambda)e^{ix} - G(\lambda)e^{-ix} \right] = r(\lambda) \sin (x + \theta(\lambda)). \]

**Proof.** Since \(e^{i\theta} = (F/G)^{1/2}\) where \((F/G)^{1/2} = 1\) at \(\lambda = 0\), it follows that \(F = re^{i\theta}\) and \(G = re^{-i\theta}\) which proves the result.

The formula (1.8) follows from Lemma 2 and 3.

If \(g(x)\) is real then (1.0) shows that \(u(x, \lambda) = u(x, \lambda)\). From this and (1.4) and (1.5) follows

\[ G(\lambda) = \overline{F(\lambda)} \quad \text{if } g \text{ is real.} \]

**Lemma 4.** The differential equation \(u'' + (1 + \lambda g)u = 0\) has for each \(\lambda\) two independent solutions \(\phi(x, \lambda)\) and \(\psi(x, \lambda)\) such that
For each $x$ they are both entire functions of $\lambda$.

**Proof.** If the solutions exist their asymptotic behavior assures their independence. To prove $\phi$ exists consider

$$
\phi(x, \lambda) = e^{ix} + \lambda \int_{x}^{\infty} \sin (x - \xi) g(\xi) \phi(\xi, \lambda) d\xi.
$$

Let $\beta(x) = \int_{x}^{\infty} |g(\xi)| d\xi$ and $\phi_0 = 0$. Then successive approximations show

$$
|\phi_{n+1} - \phi_n| \leq \frac{|\lambda|^n}{n!} (\beta(x))^n.
$$

Hence $\phi_n(x, \lambda)$ converges uniformly in $0 \leq x < \infty$ and $|\lambda| < m$, for any $m$. Hence $\phi(x, \lambda)$ exists and

$$
|\phi(x, \lambda) - e^{ix}| \leq (e^{|\lambda|\beta(x)} - 1).
$$

Since $\beta(x) \to 0$ as $x \to \infty$ this proves the result. A similar procedure holds for $\psi$.

Since $\phi$ and $\psi$ are independent it follows for each $\lambda$ that

$$
u(x, \lambda) = c_1(\lambda) \phi(x, \lambda) + c_2(\lambda) \psi(x, \lambda),$$

for some $c_1$ and $c_2$. From the asymptotic behavior of $\nu$ in Lemma 3 and of $\phi$ and $\psi$ in Lemma 4 follows

$$
c_1 = F/(2i), \quad c_2 = -G/(2i).
$$

Hence

$$
u(x, \lambda) = \frac{1}{2i} [F(\lambda) \phi(x, \lambda) - G(\lambda) \psi(x, \lambda)].
$$

If $g$ is real then $G(\lambda) = \overline{F(\lambda)}$. If $\lambda_0$ is also real and $F(\lambda_0) = 0$ then $G(\lambda_0) = \overline{F(\lambda_0)} = 0$ and hence $\nu(x, \lambda_0) = 0$ which is impossible since $\nu'(0, \lambda_0) = 1$. Thus if $g(x)$ is real then $F(\lambda)$ and $G(\lambda)$ do not vanish for real $\lambda$ and hence $\tau(\lambda)$ and $\theta(\lambda)$ are analytic for real $\lambda$. Whether $g(x)$ is real or not it follows from (2.4) that $F$ and $G$ cannot vanish simultaneously at any point $\lambda_0$ in the complex $\lambda$-plane and hence by (1.7) $\theta(\lambda)$ has a logarithmic branch point at every zero of $F$ and of $G$.

Proceeding with $J'(x, \lambda)$ much as with $J$ in Lemma 2 it follows that
\[ \lim_{x \to \infty} \left[ u'(x, \lambda) - \frac{1}{2} F(\lambda) e^{ix} - \frac{1}{2} G(\lambda) e^{-ix} \right] = 0. \]

For real \( g(x) \)
\[ u'' + (1 + \lambda g) u = 0, \quad u'' + (1 + \lambda g) \bar{u} = 0. \]

Hence
\[ \bar{u}u' - u\bar{u}' + (\lambda - \bar{\lambda}) \int_{0}^{\infty} g |u|^2 \, dx = 0. \]

Since \( u \) vanishes at \( x = 0 \), the asymptotic behavior of \( u \) and \( u' \) as \( x \to \infty \) gives
\[ |F|^2 - |G|^2 = 2i(\lambda - \bar{\lambda}) \int_{0}^{\infty} g |u|^2 \, dx. \]

Thus if \( g \geq 0 \) and \( \int_{0}^{\infty} g \, dx > 0 \),
\[ |F(\lambda)| > |G(\lambda)| \quad \text{for} \quad s\lambda < 0, \]
\[ |F(\lambda)| < |G(\lambda)| \quad \text{for} \quad s\lambda > 0. \]

(In this case since \( g \) is real \( G(\lambda) = \overline{F(\lambda)} \).) This shows that the zeros of \( G(\lambda) \) occur in the lower half-plane and those of \( F(\lambda) \) in the upper half-plane.

Reference