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ON $u''+(1+\lambda g(x))u=0$ FOR $\int_0^\infty |g(x)|\,dx<\infty$

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1. Bellman [1] has raised several questions concerning the solution $u(x, \lambda)$ of

$$u'' + [1 + \lambda g(x)]u = 0, \qquad u(0) = 0, \qquad u'(0) = 1 \tag{1.0}$$

when

$$\int_0^\infty |g(x)|\,dx < \infty. \tag{1.1}$$

He states that it is known that for real λ

$$\lim_{x\to\infty} \{u(x, \lambda) - r(\lambda) \sin [x + \theta(\lambda)]\} = 0$$

where r and θ are functions of λ. He asks for the analytic properties of r and θ if λ is a complex variable. In particular he asks whether, if $g>0$, the nearest singularity of r or θ to the origin $\lambda=0$, is on the negative real axis. It will be shown below that it is not. Indeed if g is real, r and θ are analytic functions of λ for real λ.

Let $g(x)$ be piecewise continuous, (Lebesgue integrable would suffice), and satisfy (1.1). Let

$$B(x) = \int_0^x |g(\xi)|\,d\xi. \tag{1.2}$$

THEOREM. *There is a solution* $u(x, \lambda)$ *of* (1.0) *which for each* x *is an entire function of* λ *and which satisfies*

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$$(1.3)\qquad |u(x, \lambda)| \leqq e^{|\lambda|B(x)}.$$

Let

$$(1.4)\qquad F(\lambda) = 1 - \lambda \int_0^\infty e^{-ix} g(x)u(x, \lambda)dx,$$

$$(1.5)\qquad G(\lambda) = 1 - \lambda \int_0^\infty e^{ix} g(x)u(x, \lambda)dx.$$

Then F and G are entire functions of λ *and* $F(0)=G(0)=1$. *Let*

$$(1.6)\qquad r(\lambda) = (F(\lambda)G(\lambda))^{1/2}, \qquad r(0) = 1,$$

$$(1.7)\qquad \theta(\lambda) = \frac{1}{2i} \log \frac{F(\lambda)}{G(\lambda)} \qquad \theta(0) = 0.$$

Then if λ *is not a zero of F or G*

$$(1.8)\qquad u(x, \lambda) - r(\lambda) \sin (x + \theta(\lambda)) \to 0 \qquad \text{as } x \to \infty.$$

If $g(x)$ *is real then* $G(\lambda) = \overline{F}(\bar{\lambda})$ *for all* λ *and* $F(\lambda) \neq 0$ *for real* λ. *Hence the same is true for* $G(\lambda)$ *and therefore* $r(\lambda)$ *and* $\theta(\lambda)$ *are analytic in* λ *for real* λ *and are real for real* λ. (Hence $r(\lambda)$ and $\theta(\lambda)$ have no singularities on the real axis if g is real and $r(\lambda)>0$ for real λ.) *If* $g(x) \geqq 0$ *and* $\int_0^\infty g(x)dx > 0$ *then the zeros of* $G(\lambda)$ *occur in the lower half-plane* $\mathfrak{I}\lambda < 0$ (and those of $F(\lambda) = \overline{G}(\bar{\lambda})$ *in the upper half-plane* $\mathfrak{I}\lambda > 0$).

Since F and G are entire it follows from (1.6) and (1.7) that r and θ have possible branch points at the singularities of F and G and otherwise are analytic. (The determination of r and θ at a point λ_0 where F and $G \neq 0$ is not unique but (1.8) will hold if both r and θ are obtained by an analytic continuation of (1.6) and (1.7) from $\lambda = 0$ to $\lambda = \lambda_0$ along a path that avoids zeros of F and G.)

Remark. From (1.3) and (1.4) and (1.5) it follows that nearest zero of F or G to $\lambda = 0$ is at least a distance

$$\frac{\log 2}{\int_0^\infty |g(x)| dx}$$

from $\lambda = 0$.

2. Lemma 1. *The integral equation*

$$(2.0)\qquad u(x, \lambda) = \sin x - \lambda \int_0^x \sin (x - \xi) g(\xi) u(\xi, \lambda) d\xi$$

has a solution which satisfies (1.0), *is continuous in* (x, λ) *for* $0 \leqq x < \infty$ *and* $|\lambda| < \infty$, *and entire in* λ *for each* x. *Also*

$$| u(x, \lambda) | \leqq e^{|\lambda| B(x)}.$$

Proof. Let $u_0(x, \lambda) = 0$ and use successive approximations.

Lemma 2. *Let F and G be defined by* (1.4) *and* (1.5). *Then F and G are entire functions of* λ *and*

$$u(x, \lambda) = \frac{1}{2i}\left[F(\lambda)e^{ix} - G(\lambda)e^{-ix}\right] + J \tag{2.1}$$

where

$$\lim_{x \to \infty} J(x, \lambda) = 0. \tag{2.2}$$

Proof. That F and G are entire follows from (1.1), (1.3) and (1.4) and (1.5). In (2.0) express $\sin x$ and $\sin (x - \xi)$ in exponential form and let

$$J(x, \lambda) = \lambda \int_x^\infty \sin (x - \xi) g(\xi) u(\xi, \lambda) d\xi.$$

Then (2.1) follows from (1.4) and (1.5). From (1.1) and (1.3) follows

$$| J(x, \lambda) | \leqq e^{|\lambda| B(\infty)} - e^{|\lambda| B(x)}$$

which implies (2.2).

Lemma 3. *Let* $r(\lambda) = (F(\lambda)G(\lambda))^{1/2}$, $r(0) = 1$,

$$\theta(\lambda) = (1/2i) \log F(\lambda)/G(\lambda), \theta(0) = 0.$$

Then, if λ *is not a zero of F or G*

$$\frac{1}{2i}\left[F(\lambda)e^{ix} - G(\lambda)e^{-ix}\right] = r(\lambda) \sin (x + \theta(\lambda)). \tag{2.3}$$

Proof. Since $e^{i\theta} = (F/G)^{1/2}$ where $(F/G)^{1/2} = 1$ at $\lambda = 0$, it follows that $F = re^{i\theta}$ and $G = re^{-i\theta}$ which proves the result.

The formula (1.8) follows from Lemma 2 and 3.

If $g(x)$ is real then (1.0) shows that $\bar{u}(x, \bar{\lambda}) = u(x, \lambda)$. From this and (1.4) and (1.5) follows

$$G(\lambda) = \overline{F}(\bar{\lambda}) \qquad \text{if } g \text{ is real.}$$

Lemma 4. *The differential equation* $u'' + (1 + \lambda g)u = 0$ *has for each* λ *two independent solutions* $\phi(x, \lambda)$ *and* $\psi(x, \lambda)$ *such that*

$$\lim_{x\to\infty} [\phi(x, \lambda) - e^{ix}] = 0,$$

$$\lim_{x\to\infty} [\psi(x, \lambda) - e^{-ix}] = 0.$$

For each x they are both entire functions of λ.

PROOF. If the solutions exist their asymptotic behavior assures their independence. To prove ϕ exists consider

$$\phi(x, \lambda) = e^{ix} + \lambda \int_x^\infty \sin (x - \xi) g(\xi)\phi(\xi, \lambda) d\xi.$$

Let $\beta(x) = \int_x^\infty |g(\xi)| d\xi$ and $\phi_0 = 0$. Then successive approximations show

$$|\phi_{n+1} - \phi_n| \leqq \frac{|\lambda|^n}{n!} (\beta(x))^n.$$

Hence $\phi_n(x, \lambda)$ converges uniformly in $0 \leqq x < \infty$ and $|\lambda| < m$, for any m. Hence $\phi(x, \lambda)$ exists and

$$|\phi(x, \lambda) - e^{ix}| \leqq (e^{|\lambda|\beta(x)} - 1).$$

Since $\beta(x) \to 0$ as $x \to \infty$ this proves the result. A similar procedure holds for ψ.

Since ϕ and ψ are independent it follows for each λ that

$$u(x, \lambda) = c_1(\lambda)\phi(x, \lambda) + c_2(\lambda)\psi(x, \lambda),$$

for some c_1 and c_2. From the asymptotic behavior of u in Lemma 3 and of ϕ and ψ in Lemma 4 follows

$$c_1 = F/(2i), \qquad c_2 = - G/(2i).$$

Hence

$$u(x, \lambda) = \frac{1}{2i} [F(\lambda)\phi(x, \lambda) - G(\lambda)\psi(x, \lambda)]. \tag{2.4}$$

If g is real then $G(\lambda) = \overline{F}(\bar{\lambda})$. If λ_0 is also real and $F(\lambda_0) = 0$ then $G(\lambda_0) = \overline{F}(\lambda_0) = 0$ and hence $u(x, \lambda_0) = 0$ which is impossible since $u'(0, \lambda_0) = 1$. *Thus if $g(x)$ is real then $F(\lambda)$ and $G(\lambda)$ do not vanish for real λ and hence $r(\lambda)$ and $\theta(\lambda)$ are analytic for real λ.* Whether $g(x)$ is real or not it follows from (2.4) that F and G cannot vanish simultaneously at any point λ_0 in the complex λ-plane and hence by (1.7) $\theta(\lambda)$ has a logarithmic branch point at every zero of F and of G.

Proceeding with $J'(x, \lambda)$ much as with J in Lemma 2 it follows that

$$\lim_{x\to\infty}\left[u'(x,\lambda)-\frac{1}{2}F(\lambda)e^{ix}-\frac{1}{2}G(\lambda)e^{-ix}\right]=0.$$

For real $g(x)$

$$u''+(1+\lambda g)u=0, \qquad \bar{u}''+(1+\bar{\lambda}g)\bar{u}=0.$$

Hence

$$\bar{u}u'-u\bar{u}'\Big]_0^\infty+(\lambda-\bar{\lambda})\int_0^\infty g\,|\,u\,|^2dx=0.$$

Since u vanishes at $x=0$, the asymptotic behavior of u and u' as $x\to\infty$ gives

$$|\,F\,|^2-|\,G\,|^2=2i(\lambda-\bar{\lambda})\int_0^\infty g\,|\,u\,|^2dx.$$

Thus if $g\geqq 0$ and $\int_0^\infty g dx>0$,

$$|\,F(\lambda)\,| > |\,G(\lambda)\,| \quad \text{for} \quad \mathcal{I}\lambda<0,$$
$$|\,F(\lambda)\,| < |\,G(\lambda)\,| \quad \text{for} \quad \mathcal{I}\lambda>0.$$

(In this case since g is real $G(\lambda)=\overline{F}(\bar{\lambda})$.) This shows that the zeros of $G(\lambda)$ occur in the lower half-plane and those of $F(\lambda)$ in the upper half-plane.

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