ON GENERALIZED GROUP ALGEBRAS[1]

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1. **Introduction.** Let $G=\{a, b, c, \cdots\}$ denote an arbitrary locally compact abelian group and let $X=\{x, y, z, \cdots\}$ be any complex commutative Banach algebra with identity e. The generalized group algebra $B(G, X)$ is the set of all strongly measurable, Bochner integrable functions[2] defined on G with values in X. Functions which differ only on sets of Haar measure zero are to be identified. In $B(G, X)$ we define addition, scalar multiplication and multiplication (convolution) as follows: $(f+g)(a)=f(a)+g(a)$, $(\lambda f)(a)=\lambda f(a)$ where λ is a complex number and $(f*g)(a)=\int_G f(b)g(a-b)db$. The latter integral is taken in the sense of Bochner with respect to Haar measure in G. With $\|f\|_B=\int_G |f(a)|_X da$ as norm, $B(G, X)$ can be shown to be a complex commutative Banach algebra. If X reduces to the complex numbers, then $B(G, X)$ specializes into the classical group algebra $L(G)$.

In §2 of this paper we will be concerned with bounded representations of $B(G, X)$ by operators in a Hilbert space H. Following Theorem 1, we give an extension of the Stone-Ambrose-Godement theorem on unitary representations of locally compact abelian groups. In §3, kernels and hulls will be studied; applications to the case of the classical group algebra $L(G_1\times G_2)$ of the topological product of two groups will be given.

2. **Representations of** $B(G, X)$. If f is in $B(G, X)$ or in $L(G)$ and if $x\in X$, then fx shall denote the function $(fx)(a)=f(a)x$ for $a\in G$. It is clear that $fx\in B(G, X)$.

We may suppose, without limiting the generality, that the identity e in X is such that $|e|=1$. Define $Le=\{ge \mid g\in L(G)\}$. Then Le is contained in $B(G, X)$ and Le is isometrically isomorphic with $L(G)$.

The convolution $f*g=\int f(b)g(a-b)db$ of a function $f\in B$ with a function $g\in L$ gives a function in B. (All integrals will always be taken over G unless otherwise indicated.)

For the definition of a $*$-representation and self-adjoint algebra

Received by the editors December 5, 1957, and, in revised form, June 2, 1958.

[1] This paper constitutes a revised version of a portion of the author's Yale (1956) doctoral dissertation. Some results used in this work are to be found in [2i]. The reader should also see [2ii] and [5] in the references at the end.

[2] For the theory and properties of Bochner and Pettis integrals used in this paper, see [4, pp. 40–50].

the reader should see [6] or [7]. If $x \to x^*$ is an involution in X, then $f(a) \to \bar{f}(a) \equiv f^*(-a)$ is an involution in B.

We may now enunciate

THEOREM 1. *Let $\{A_f\}$ be a representation of $B(G, X)$ by operators of bound K in a complex Hilbert space H, i.e., $\|A_f\| \leqq K\|f\|_B$. Suppose $\{A_f\}$ is a $*$-representation on $Le \subset B(G, X)$ and suppose the union of the ranges of the A_f's with $f \in Le$ is a dense in H. Then there exists a representation $\{T_x\}$ of bound K of the algebra X in H and a weakly continuous unitary representation $\{U_a\}$ of G in H such that every T_x commutes with every U_a and $A_f\phi = \int T_{f(a)} U_a \phi da$ for each $f \in B$, $\phi \in H$. This integral exists in the sense of Pettis, i.e., $(A_f\phi, \psi) = \int (T_{f(a)} U_a \phi, \psi) da$ for all $\psi \in H$.*

Conversely, if $\{U_a\}$, $\{T_x\}$ are respectively a weakly continuous unitary representation of G in H and a bounded representation of X in H, and if each T_x commutes with each U_a, then there exists a bounded representation $\{A_f\}$ of $B(G, X)$ in H such that $A_f\phi$ is given by the Pettis integral $\int T_{f(a)} U_a \phi da$ for $\phi \in H$. If, furthermore, X is self-adjoint, then $\{A_f\}$ is a $*$-representation if $\{T_x\}$ is a $*$-representation.

PROOF. Let $\{j_W\}$ be an approximate identity in $L(G)$, i.e., for each neighborhood W of $0 \in G$, j_W is a (numerical) non-negative function vanishing off W such that $\int j_W(a) da = 1$. Using a method exactly like that in [6, p. 128], it may be shown that $A_{j_W x}$ converges strongly, as $W \to 0$, to an operator T_x in H such that $\|T_x\| \leqq K|x|$ for each $x \in X$. Furthermore $\{T_x\}$ is easily seen to be a representation of X in H and $A_{fx} = T_x A_f = A_f T_x$ for all $x \in X$, $f \in B$. By hypothesis $\{A_f\}$ is a $*$-representation on $Le \subset B(G, X)$ with dense range in H, so there exists a weakly continuous representation $\{U_a\}$ of G in H such that $(A_{fe}\phi, \psi) = \int f(a)(U_a\phi, \psi) da$ for every $\phi, \psi \in H$ and $f \in L(G)$ (see [6, pp. 128–129]). The representation $\{U_a\}$ actually turns out to be strongly continuous, but we make no use of this fact here. The U_a, as their construction shows, are strong limits of A_f's and, as such, they satisfy $U_a T_x = T_x U_a$ for every $x \in X$, $a \in G$. If $f \in L(G)$, $x \in X$, ϕ and $\psi \in H$, we have: $(A_{fe} T_x \phi, \psi) = (A_{fx}\phi, \psi) = \int f(a)(U_a T_x \phi, \psi) da = \int (f(a) U_a T_x \phi, \psi) da = \int (T_{f(a)x} U_a \phi, \psi) da$. Hence $A_{fx}\phi$ is given by the Pettis integral $\int T_{f(a)x} U_a \phi da$ because H is its own conjugate space.

Suppose now that g is an arbitrary function in $B(G, X)$. Since the finite linear combinations of functions of the type fx with $f \in L(G)$, $x \in X$ are dense in $B(G, X)$, we have a sequence $g_n \to g$ in $B(G, X)$-norm such that $(A_{g_n}\phi, \psi) = \int (T_{g_n(a)} U_a \phi, \psi) da$ and $(A_{g_n}\phi, \psi) \to (A_g\phi, \psi)$ as $n \to \infty$. We show that $(A_g\phi, \psi) = \int (T_{g(a)} U_a \phi, \psi) da$ and this will establish the first part of the theorem. We have:

$$\left| \int (T_{g_n(a)} U_a\phi, \psi) da - \int (T_{g(a)} U_a\phi, \psi) da \right|$$

$$\leqq \int \left| ([T_{g_n(a)} - T_{g(a)}] U_a\phi, \psi) \right| da$$

$$\leqq \int \|T_{g_n(a)} - T_{g(a)}\| \cdot \|U_a\| \cdot |\phi| \cdot |\psi| \, da \leqq K\|g_n - g\|_B \cdot |\phi| \cdot |\psi| \to 0$$

as $n \to \infty$.

We turn to the converse of the first part of the theorem. For any $f \in B(G, X)$ there is a sequence $\{g_n\}$ of simple functions converging to f strongly, almost everywhere. Each function $(T_{g_n(a)} U_a\phi, \psi)$, for any $\phi, \psi \in H$, is measurable in a because $\{U_a\}$ is weakly continuous. Since $(T_{g_n(a)} U_a\phi, \psi) \to (T_{f(a)} U_a\phi, \psi)$ a.e. as $n \to \infty$, it follows that $(T_{f(a)} U_a\phi, \psi)$ is measurable in a being the pointwise limit of measurable functions.

Now, since $(T_{f(a)} U_a\phi, \psi)$ is measurable, we may speak of the integral $\int (T_{f(a)} U_a\phi, \psi) da$. This may be viewed as a bounded bilinear functional in H for each $f \in B$, for $\left| \int (T_{f(a)} U_a\phi, \psi) da \right| \leqq K\|f\|_B \cdot |\phi| \cdot |\psi|$. Hence there exists a unique operator A_f in H, for each $f \in B$, such that $(A_f\phi, \psi) = \int (T_{f(a)} U_a\phi, \psi) da$ and $\|A_f\| \leqq K\|f\|_B$. We show $\{A_f\}$ is a representation of $B(G, X)$. It is clear that $A_{f+g} = A_f + A_g$ and $A_{\lambda f} = \lambda A_f$. To show $A_{f*g} = A_f A_g$, we first observe that $(T_{\int f(a) da}\phi, \psi) = \int (T_{f(a)}\phi, \psi) da$ for any $f \in B$ and $\phi, \psi \in H$. This latter equation is true for simple functions and the general result follows from the usual approximation argument. Therefore, for any $f, g \in B$ and $\phi, \psi \in H$, $(T_{f*g(a)}\phi, \psi) = \int (T_{f(b)} T_{g(a-b)}\phi, \psi) db$ a.e. in a. Now:

$$\begin{aligned}
(A_{f*g}\phi, \psi) &= \int (T_{f*g} U_a\phi, \psi) da = \int\int (T_{f(b)} T_{g(a-b)} U_a\phi, \psi) db da \\
&= \int\int (T_{f(b)} T_{g(a-b)} U_a\phi, \psi) da db \\
&= \int\int (T_{f(b)} T_{g(a)} U_{a+b}\phi, \psi) da db \\
&= \int\int (U_a T_{g(a)}\phi, (U_b T_{f(b)})^* \psi) da db \\
&= \int (U_b T_{f(b)} A_g\phi, \psi) db = (A_f A_g\phi, \psi).
\end{aligned}$$

Since ϕ, ψ are arbitrary, it follows that $A_{f*g} = A_f A_g$. In several steps,

in the above computation, we use the assumption that the T_x commute with the U_a and, in one step, we employed the Fubini theorem.

The last part of the theorem remains to be established. Suppose $\{T_x\}$ is a $*$-representation of X in H. Then $A_{\bar{f}}=A_f^*$ as we show:

$$\begin{aligned}(A_{\bar{f}}\phi, \psi) &= \int (T_{\bar{f}(a)}U_a\phi, \psi)da = \int (T_{f^*(-a)}U_a\phi, \psi)da \\ &= \int (T^*_{f(-a)}U_a\phi, \psi)da = \int (T^*_{f(a)}U^*_a\phi, \psi)da \\ &= \int ((U_aT_{f(a)})^*\phi, \psi)da = \left[\int (U_aT_{f(a)}\psi, \phi)da\right]^* \\ &= (A_f\psi, \phi)^* = (A_f^*\phi, \psi).\end{aligned}$$

Hence $A_{\bar{f}}=A_f^*$ and this completes the proof.

Let $\mathfrak{M}(X)$ denote the space of maximal ideals in X in the usual weak topology and let $\hat{G}$ denote the dual (character) group of G in the Pontrjagin topology. Suppose ϕ_M is the canonical homomorphism of X onto the complex numbers associated with $M\in\mathfrak{M}(X)$ and define the "Fourier transform" of $f\in B$ as $\hat{f}(M, \hat{a})=\int\phi_M f(a)(a, \hat{a})da$. The transform $\hat{f}$ is a function of both M and $\hat{a}$. It has been shown [2i] that each continuous multiplicative linear functional in $B(G, X)$ is of the "Fourier transform" type and, further, $\mathfrak{M}(B)$ (the space of regular maximal ideals in $B(G, X)$) is homeomorphic with the topological product of $\mathfrak{M}(X)$ and $\hat{G}$. We use these results from [2i] in proving

THEOREM 2 (GENERALIZED STONE THEOREM). *Suppose X is a self-adjoint algebra and $\{T_x\}$, $\{U_a\}$, G and H have the meanings of Theorem 1. Let $\{T_x\}$ be a $*$-representation commuting with $\{U_a\}$. Then there exists a projection-valued measure $P(M, \hat{a})$ on $\mathfrak{M}(X)\times\hat{G}$ (see above) such that*

$$(T_xU_a\phi, \psi) = \int_{\mathfrak{M}(X)\times\hat{G}} \phi_M(x)(a, \hat{a})d(P(M, \hat{a})\phi, \psi)$$

for all $a\in G$, $x\in X$, ϕ and $\psi\in H$.

PROOF. Since $\{T_x\}$ is a $*$-representation of X, $\{T_x\}$ is a bounded representation. By Theorem 1, $A_f\phi=\int T_{f(a)}U_a\phi da$ defines a $*$-representation of $B(G, X)$ in H. It is known, [7, Theorem 57], that if $\{S_y\}$ is a $*$-representation of a self-adjoint algebra Y by operators in a Hilbert space H, then there exists a projection-valued measure $P(M)$ defined on the space $\mathfrak{M}(Y)$ of regular maximal ideals in Y

such that $(S_y\phi, \psi)=\int_{\mathfrak{M}(Y)}\phi_M(y)d(P(M)\phi, \psi)$ for $\phi, \psi\in H$. Taking $Y=B(G, X)$ and $\{S_y\}$ to be $\{A_f\}$, we have a projection-valued measure $P(M, \hat{a})$ defined over $\mathfrak{M}(X)\times\hat{G}$ such that

$$(A_f\phi, \psi) = \int (T_{f(a)}U_a\phi, \psi)da = \int^{-} \hat{f}(M, \hat{a})d(P(M, \hat{a})\phi, \psi).$$

($\int^{-}$ means integrate over $\mathfrak{M}(X)\times\hat{G}$.)

Suppose $g\in L(G)$, $x\in X$. Then

$$\begin{aligned}(A_{gx}\phi, \psi) &= \int g(a)(T_xU_a\phi, \psi)da \\ &= \int^{-}\left(\int g(a)\phi_M(x)(a, \hat{a})da\right)d(P(M, \hat{a})\phi, \psi) \\ &= \int g(a)\left(\int^{-}\phi_M(x)(a, \hat{a})d(P(M, \hat{a})\phi, \psi)\right)da.\end{aligned}$$

Since g is arbitrary in $L(G)$ it follows that

$$(T_xU_a\phi, \psi) = \int^{-}\phi_M(x)(a, \hat{a})d(P(M, \hat{a})\phi, \psi).$$

This ends the proof.

We conclude this section with two observations. First, if G reduces to the one-point group $\{0\}$ and X is arbitrary, then $\{U_a\}$ reduces to the single identity operator I in H and the above result becomes $(T_x\phi, \psi)=\int_{\mathfrak{M}(X)}\phi_M(x)d(P(M)\phi, \psi)$ which is precisely Theorem 57 in [7]. Second, if G is arbitrary while X reduces to the algebra of complex numbers and $T_x=xI$, then

$$\begin{aligned}(xU_a\phi, \psi) &= x(U_a\phi, \psi) = \int_{\hat{G}} x(a, \hat{a})d(P(\hat{a})\phi, \psi) \\ &= x\int_{\hat{G}}(a, \hat{a})d(P(\hat{a})\phi, \psi)\end{aligned}$$

so that

$$(U_a\phi, \psi) = \int_{\hat{G}}(a, \hat{a})d(P(\hat{a})\phi, \psi).$$

Our theorem thus specializes into Stone's theorem.

3. **Kernels and hulls.** In this section we shall be concerned with proving a few results on kernels and hulls in $B(G, X)$. We will apply

these to the (classical) group algebra $L(G_1 \times G_2)$ over the topological product of two locally compact abelian groups. We do not assume that X possesses an identity except in Theorem 4.

We recall that if X is a B-algebra and A is a closed ideal in X, then the hull of A, $h(A)$, is the set of all regular maximal ideals containing A. Let $k(h(A))$ be the kernel of $h(A)$, i.e., the intersection of all ideals in $h(A)$. An ideal A is said to be "the kernel of its hull" if $A = k(h(A))$.

If A is an ideal in X then $I(A)$ shall denote

$$\{f \in B(G, X) \mid f(a) \in A \text{ a.e. over } G\}.$$

If A is a proper closed ideal in X, then $I(A)$ is a proper closed ideal in $B(G, X)$ as is easy to show.

As indicated in §2, each $(M, \hat{a}) \in \mathfrak{M}(X) \times \hat{G}$ gives rise to a multiplicative linear functional in B. We shall denote the regular maximal ideal in B which is the kernel of this functional by $I(M, \hat{a})$.

THEOREM 3. (i) *If A is a proper closed ideal in X, then* $h(I(A)) = \{I(M, \hat{a}) \mid M \in h(A), \hat{a} \in \hat{G}\}$.

(ii) $k(hI(A)) = I(A)$ *if and only if* $k(h(A)) = A$.

PROOF. (i) Suppose $f \in I(A)$ and $M \in h(A)$. Then for any $\hat{a} \in \hat{G}$ we have $\int \phi_M f(a)(a, \hat{a})da = 0$ so that $I(M, \hat{a}) \in h(I(A))$.

Assume that $I(N, \hat{b}) \supset I(A)$ for some $(N, \hat{b}) \in \mathfrak{M}(X) \times \hat{G}$. We wish to show that $N \in h(A)$. Let $f = gx$ with g chosen in $L(G)$ such that $\hat{g}(\hat{b}) = \int g(a)(a, \hat{b})da \neq 0$ and with $x \in A - \{0\}$. Clearly $gx \in I(A)$. Now, $\int \phi_N(gx)(a)(a, \hat{b})da = \int \phi_N(x)g(a)(a, \hat{b})da = \hat{g}(\hat{b})\phi_N(x) = 0$. This means $\phi_N(x) = 0$ for each $x \in A$. Hence $N \supset A$ and $N \in h(A)$.

(ii) Suppose $A = k(h(A))$. If $f \in k(hI(A))$, then $\hat{f}(M, \hat{a}) = 0$ for all $M \in h(A)$ and all $\hat{a} \in \hat{G}$. Since $L(G)$ is semi-simple, $\phi_M f = 0$ a.e. over G for all $M \in h(A)$. This means $f \in k(h(A))$ a.e. over G, as we proceed to show. If $\{j_W\}$ is an approximate identity in $L(G)$ consisting of bounded functions, then $j_W * f$ is continuous from G to X since f is continuous in B-norm. Hence $j_W * f \in k(h(A))$ *everywhere* over G. Let $\{j_{W_n}\}$ be a sequence chosen from $\{j_W\}$ such that $j_{W_n} * f \to f$ in B-norm. There exists a subsequence $\{j_{W_{nk}}\}$ in $\{j_{W_n}\}$ such that $j_{W_{nk}} * f \to f$ pointwise a.e. in X-norm. Hence $f \in k(h(A))$ a.e. over G.

Suppose, to prove the converse, that $k(h(A)) \neq A$. There exists an $x \in k(h(A))$ such that $x \notin A$. The function gx, with $g \neq 0$ in $L(G)$, is in $k(hI(A))$ but not in $I(A)$.

COROLLARY. (1) *The only regular maximal ideals containing $I(M)$ with $M \in \mathfrak{M}(X)$ are of the form $I(M, \hat{a})$ with $\hat{a} \in \hat{G}$. Also $I(M)$ is the kernel of its hull.*

(2) *If A is a closed ideal in X not contained in any regular maximal ideal, then $I(A)$ is not contained in any regular maximal ideal in $B(G, X)$.*

(3) *There exists an ideal in $B(G, X)$ which is not the kernel of its hull if this is true of the algebra X.*

(4) *If $M, N \in \mathfrak{M}(X)$ with $M \neq N$ and if $\hat{a} \in \hat{G}$, then $I(M, \hat{a}) + I(N) = B(G, X)$.*

PROOF. (1) follows from Theorem 3 (i), (ii). (2) and (3) follow from Theorem 3, parts (i) and (ii), respectively. To prove (4), we observe that $J \equiv I(M, \hat{a}) + I(N)$ is an ideal containing $I(M, \hat{a})$ and $I(N)$. This means $J = I(M, \hat{a})$ or $J = B(G, X)$. If $J = I(M, \hat{a})$, then $I(M, \hat{a}) \supset I(N)$ which is impossible by (1) of the corollary. Hence $J = B(G, X)$.

We apply the above results to the algebra $L(G_1 \times G_2)$ where $G_1 = \{a\}$, $G_2 = \{b\}$ are two locally compact abelian groups.[3] If $f \in L(G_1 \times G_2)$ and if $a \in G_1$, then the function f_a defined over G_2 by $f_a(b) = f(a, b)$ is designated as the section of f determined by a. Since $f \in L(G_1 \times G_2)$, every section of f is measurable over G_2 and almost every section f_a is in $L(G_2)$ [1, p. 148]. The map $\pi: L(G_1 \times G_2) \to B(G_1, L(G_2))$ defined as $(\pi f)(a) = f_a(\cdot)$ can be easily shown to be an isometric isomorphism. By means of the map π, $L(G_1 \times G_2)$ may be viewed as an algebra $B(G, X)$ with $G = G_1$ and $X = L(G_2)$. Hence we may interpret our preceding results in the language of $L(G_1 \times G_2)$. The ideals $I(M)$ then become ideals $I(\hat{b})$ consisting of those functions $f \in L(G_1 \times G_2)$ such that $\int_{G_2} f(a, b)(b, \hat{b}) db = 0$ a.e. in a for a fixed $\hat{b} \in \hat{G}_2$. The $I(M, \hat{a})$ become $I(\hat{b}, \hat{a})$. These consist of those functions $f \in L(G_1 \times G_2)$ such that $\iint_{G_1 \times G_2} f(a, b)(a, \hat{a})(b, \hat{b}) da db = 0$.

Corollary 1 to Theorem 3 states that the ideals $I(\hat{b})$ in $L(G_1 \times G_2)$ are the kernels of their hulls.

If G_2 is taken to be the additive group of Euclidean n-space E_n $(n \geqq 3)$, then L. Schwartz has shown [8] that there are closed ideals in $L(E_n)$ which are smaller than the kernels of their hulls. Hence Theorem 3, Corollary 3 says that the group algebras $L(G \times E_n)$ have closed ideals smaller than the kernels of their hulls for *any* abelian group G when $n \geqq 3$.

As a further application we see that $I(\hat{b}_1, \hat{a}) + I(\hat{b}_2) = L(G_1 \times G_2)$ for any $\hat{a} \in \hat{G}_1$ and $\hat{b}_1, \hat{b}_2 \in \hat{G}_2$ with $\hat{b}_1 \neq \hat{b}_2$, by (4) of the Corollary.

We shall return to $L(G_1 \times G_2)$ after some more results about $B(G, X)$ are established. First we observe that the Segal-Kaplansky-

[3] These applications were suggested by Professor J. T. Schwartz. For a discussion of group algebras of the type $L(G_1 \times G_2)$ with G_1 compact but not necessarily abelian and G_2 locally compact and abelian see [9].

Helson theorem [3; 6] is, in general, no longer true for $B(G, X)$. This theorem states that a closed ideal in the group algebra $L(G)$ is the kernel of its hull if the hull has a boundary in $\hat{G}$ containing no non-null perfect set. As a counter-example for an algebra $B(G, X)$, let $G=\{0\}$ be the one-point group; then $\hat{G}=\{\hat{0}\}$ and $(0, \hat{0})=1$. Take X to be a B-algebra in which there is a closed primary ideal A, i.e., A is contained in and smaller than precisely one regular maximal ideal M_0. Then $h(I(A))=\{I(M_0, \hat{0})\}$ and $I(A)$ thus has a one-point hull. Yet $k(hI(A))\neq I(A)$ by Theorem 3 (ii). We may even take X in this example to be semi-simple and regular so that these conditions are not sufficient for a general theorem. We can, however, prove:

THEOREM 4. *Let X have an identity e and let I be a closed ideal in $B(G, X)$ with $h(I)=\{I(M_i, \hat{a}_i) \mid i\in\Omega\}$. Suppose the set $\{\hat{a}_i \mid i\in\Omega\}$ has a boundary in $\hat{G}$ which contains no non-null perfect set and suppose $I\supset I(M_0)$ for some $M_0\in\mathfrak{M}(X)$. Then $k(h(I))=I$.*

PROOF. Since $I\supset I(M_0)$ for some $M_0\in\mathfrak{M}(X)$, we have $M_i=M_0$ for all $i\in\Omega$ by Theorem 3, Corollary (1). Therefore $h(I)=\{I(M_0, \hat{a}_i) \mid i\in\Omega\}$. Define $K=\{f\in B(G, X) \mid f\in I\cap Le\}$. K is seen to be a closed ideal in Le and it may be shown that $h(K)$ (the hull of K in the B-algebra Le) is $\{\hat{a}_i \mid i\in\Omega\}$. By hypothesis, the boundary of $h(K)$ contains no non-null perfect set and this means $K=k(h(K))$ by the Segal-Kaplansky-Helson theorem. This, in turn, implies $I=k(h(I))$. For, let $f\in k(h(I))$. Then $\hat{f}(M_0, \hat{a}_i)=0$ for all $\hat{a}_i\in h(K)$ and so $(\phi_{M_0}f)e\in K$. Since $f-(\phi_{M_0}f)e\in I(M_0)$ and $I\supset I(M_0)$, we have $f=(\phi_{M_0}f)e+[f-(\phi_{M_0}f)e]\in I$.

The next theorem will pertain to compact groups and will require the following lemma.

LEMMA. *Suppose $\phi_M f$ is constant a.e. for each $M\in\mathfrak{M}(X)$ where $f\in B(G, X)$ with G compact and X semi-simple. Then f is constant a.e.*

PROOF. Since $\phi_M f$ is constant a.e. and since $\int(a, \hat{a})da=0$ if $\hat{a}\neq\hat{0}$, we have $\hat{f}(M, \hat{a})=0$ if $\hat{a}\neq\hat{0}$ for all $M\in\mathfrak{M}(X)$. If the Haar measure of G is normalized to 1, then $\hat{f}(M, \hat{a})-\hat{f}(M, \hat{0})\cdot\int(a, \hat{a})da=0$ for *all* $(M, \hat{a})$ because $\int(a, \hat{0})da=1$. This means $f-\int f(a)da$ has a Fourier transform vanishing everywhere over $\mathfrak{M}(X)\times\hat{G}$. Thus $f=\int f(a)da$ a.e. over G because $B(G, X)$ is semi-simple if X is semi-simple [2i].

THEOREM 5. *Suppose G is a compact group (not the one-point group). Then X is isometrically and isomorphically contained in $B(G, X)$ and X is a closed ideal in $B(G, X)$. Further, X is the kernel of its hull if and only if X is semi-simple as a B-algebra.*

PROOF. We may consider the B-algebra X as contained as a subset of $B(G, X)$ by identifying each $x \in X$ with the constant mapping $G \to x$. The elements of X can thus be viewed as constants in $B(G, X)$. When the Haar measure m of G is normalized so that $m(G)=1$, then X is isometrically and isomorphically contained in $B(G, X)$ as is readily seen. If $g \in B$, $x \in X$, then $g * x = \hat{g}(\hat{0})x$ so that X is a closed ideal in B.

Now $I(M, \hat{a}) \supset X$ for all $(M, \hat{a})$ with $\hat{a} \neq \hat{0}$. We will prove that $I \equiv \bigcap I(M, \hat{a})$, where the intersection is taken over $\mathfrak{M}(X) \times (\hat{G} - \{\hat{0}\})$, is identical with X. If $f \in I$, then $\hat{f}(M, \hat{a}) = 0$ for all $\hat{a} \neq \hat{0}$ so that $\phi_M f$ is a constant a.e. for all $M \in \mathfrak{M}(X)$. Since X is semi-simple, f is constant a.e. by the preceding lemma. X is therefore the kernel of its hull.

Suppose X is not semi-simple. If f is not a constant function in $L(G)$ (such a function exists if G is not the one-point group) and if $x(\neq 0)$ is in all the regular maximal ideals of X, then $fx \in I$ (see above), but $fx \notin X$.

We conclude with more applications to $L(G_1 \times G_2)$. If G_2 is taken to be a discrete group, then $L(G_2)$ has an identity and Theorem 4 is applicable. We construct an ideal I in $L(G_1 \times G_2)$ which meets the requirements of this theorem. Suppose $\{\hat{a}_i\}$ is a set in $\hat{G}_1$ with boundary containing no non-null perfect set. For example, $\{\hat{a}_i\}$ may be taken finite. Let f be a function in $L(G_1)$ whose Fourier transform vanishes precisely at the set $\{\hat{a}_i\}$. Suppose $\hat{b}_1, \hat{b}_2 \in \hat{G}_2$ with $\hat{b}_1 \neq \hat{b}_2$. Choose $g \in L(G_2)$ such that $\hat{g}(\hat{b}_1) \neq 0$, $\hat{g}(\hat{b}_2) = 0$. Let I be the closed ideal generated by $I(\hat{b}_1)$ and the function $f(\cdot)g(\cdot) \in L(G_1 \times G_2)$, i.e., I consists of all limits of sequences of functions of the type $f(\cdot)g(\cdot) * r(\cdot, \cdot) + s(\cdot, \cdot)$ with $r \in L(G_1 \times G_2)$ and $s \in I(\hat{b}_1)$. The hull of I is $\{I(\hat{b}_1, \hat{a}_i)\}$ as is easy to show. I is therefore the kernel of its hull by Theorem 4.

Now consider $L(G_1 \times G_2)$ with G_1 a compact group and G_2 arbitrary. Since $L(G_2)$ is semi-simple, Theorem 5 says that the functions $f \in L(G_1 \times G_2)$, which are independent of $a \in G_1$, form a closed ideal which is the kernel of its hull.

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A NOTE ON DERIVATIONS OF LIE ALGEBRAS II

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Let L be a Lie algebra over a ground field of characteristic 0 and let $D(L)$ and $I(L)$ denote the Lie algebra of derivations and inner derivations of L respectively.

The following theorem is proved, although not explicitly stated, in an earlier note.[1]

THEOREM 1. *Let $L=S+R$ (Levi decomposition) and let $\mathfrak{A}(S)=\{D \mid D\in D(L),\ D(S)=(0)\}$. Further let ρ denote the restriction homomorphism of $D(L)$ into $D(R)$. Then $D(L)$ splits over $I(L)\rightleftarrows\rho(\mathfrak{A}(S))$ splits over $\rho(\mathfrak{A}(S))\cap I(R)$.*

If S is a semi-simple Lie algebra and M is any S-module then M is the direct sum of M^S and $S\cdot M$ where M^S is the trivial submodule of M.

THEOREM 2. *Let $L=S+R$ (Levi decomposition). If $R^S\subseteq Z(R)$ (the center of R) then $D(L)$ splits over $I(L)$.*[2]

PROOF. Let $u\in R$ and suppose that the derivation of R that is effected by u is the restriction to R of a derivation of L that annihilates S. Then $[S, u]\subseteq Z(R)$. As an S-module, R is the direct sum of $Z(R)$ and a complementary submodule P. The component of u in P is annihilated by S so that $u\in R^S+Z(R)\subseteq Z(R)$. This means

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Received by the editors May 17, 1958 and, in revised form, June 3, 1958.

[1] Details will be found in G. F. Leger, *A note on the derivations of Lie algebras*, Proc. Amer. Math. Soc. vol. 4 (1953). We refer to this note as I.

[2] The author is grateful to the referee for putting Theorem 2 in this form.