ON GENERALIZED GROUP ALGEBRAS

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1. Introduction. Let $G = \{a, b, c, \cdots \}$ denote an arbitrary locally compact abelian group and let $X = \{x, y, z, \cdots \}$ be any complex commutative Banach algebra with identity $e$. The generalized group algebra $B(G, X)$ is the set of all strongly measurable, Bochner integrable functions defined on $G$ with values in $X$. Functions which differ only on sets of Haar measure zero are to be identified. In $B(G, X)$ we define addition, scalar multiplication and multiplication (convolution) as follows: $(f+g)(a) = f(a) + g(a)$, $(\lambda f)(a) = \lambda f(a)$ where $\lambda$ is a complex number and $(f*g)(a) = \int_a f(b)g(a-b)db$. The latter integral is taken in the sense of Bochner with respect to Haar measure in $G$. With $\|f\|_B = \int_G |f(a)| xda$ as norm, $B(G, X)$ can be shown to be a complex commutative Banach algebra. If $X$ reduces to the complex numbers, then $B(G, X)$ specializes into the classical group algebra $L(G)$.

In §2 of this paper we will be concerned with bounded representations of $B(G, X)$ by operators in a Hilbert space $H$. Following Theorem 1, we give an extension of the Stone-Ambrose-Godement theorem on unitary representations of locally compact abelian groups. In §3, kernels and hulls will be studied; applications to the case of the classical group algebra $L(G_1 \times G_2)$ of the topological product of two groups will be given.

2. Representations of $B(G, X)$. If $f$ is in $B(G, X)$ or in $L(G)$ and if $x \in X$, then $fx$ shall denote the function $(fx)(a) = f(a)x$ for $a \in G$. It is clear that $fx \in B(G, X)$.

We may suppose, without limiting the generality, that the identity $e$ in $X$ is such that $|e| = 1$. Define $Le = \{ge : g \in L(G)\}$. Then $Le$ is contained in $B(G, X)$ and $Le$ is isometrically isomorphic with $L(G)$.

The convolution $f*g = \int f(b)g(a-b)db$ of a function $f \in B$ with a function $g \in L$ gives a function in $B$. (All integrals will always be taken over $G$ unless otherwise indicated.)

For the definition of a $*$-representation and self-adjoint algebra
the reader should see [6] or [7]. If \( x \to x^* \) is an involution in \( X \), then \( f(a) \to f(a)^* = f^*(-a) \) is an involution in \( B \).

We may now enunciate

**Theorem 1.** Let \( \{A_f\} \) be a representation of \( B(G, X) \) by operators of bound \( K \) in a complex Hilbert space \( H \), i.e., \( \|A_f\| \leq K\|f\|_B \). Suppose \( \{A_f\} \) is a \(*\)-representation on \( L_e \subset B(G, X) \) and suppose the union of the ranges of the \( A_f \)'s with \( f \in L_e \) is a dense in \( H \). Then there exists a representation \( \{T_x\} \) of bound \( K \) of the algebra \( X \) in \( H \) and a weakly continuous unitary representation \( \{U_\alpha\} \) of \( G \) in \( H \) such that every \( T_x \) commutes with every \( U_\alpha \) and \( A_\phi = \int T_{f(a)} U_\alpha \phi da \) for each \( f \in B, \phi \in H \). This integral exists in the sense of Pettis, i.e., \( (A_\phi, \psi) = \int (T_{f(a)} U_\alpha \phi, \psi) da \) for all \( \psi \in H \).

Conversely, if \( \{U_\alpha\}, \{T_x\} \) are respectively a weakly continuous unitary representation of \( G \) in \( H \) and a bounded representation of \( X \) in \( H \), and if each \( T_x \) commutes with each \( U_\alpha \), then there exists a bounded representation \( \{A_f\} \) of \( B(G, X) \) in \( H \) such that \( A_\phi \) is given by the Pettis integral \( \int T_{f(a)} U_\alpha \phi da \) for \( \phi \in H \). If, furthermore, \( X \) is self-adjoint, then \( \{A_f\} \) is a \(*\)-representation if \( \{T_x\} \) is a \(*\)-representation.

**Proof.** Let \( \{j_\omega\} \) be an approximate identity in \( L(G) \), i.e., for each neighborhood \( W \) of \( 0 \in G \), \( j_\omega \) is a (numerical) non-negative function vanishing off \( W \) such that \( \int j_{\omega x} da = 1 \). Using a method exactly like that in [6, p. 128], it may be shown that \( A_{j_\omega x} \) converges strongly, as \( \omega \to 0 \), to an operator \( T_x \) in \( H \) such that \( \|T_x\| \leq K|x| \) for each \( x \in X \). Furthermore \( \{T_x\} \) is easily seen to be a representation of \( X \) in \( H \) and \( A_{j_\omega x} = T_x A_{j_\omega} = A_f T_x \) for all \( x \in X, f \in B \). By hypothesis \( \{A_f\} \) is a \(*\)-representation on \( L_e \subset B(G, X) \) with dense range in \( H \), so there exists a weakly continuous representation \( \{U_\alpha\} \) of \( G \) in \( H \) such that \( (A_\phi, \psi) = \int (T_{f(a)} U_\alpha \phi, \psi) da \) for every \( \phi, \psi \in H \) and \( f \in L(G) \) (see [6, pp. 128–129]). The representation \( \{U_\alpha\} \) actually turns out to be strongly continuous, but we make no use of this fact here. The \( U_\alpha \), as their construction shows, are strong limits of \( A_{j_\omega} \)'s and, as such, they satisfy \( U_\alpha T_x = T_x U_\alpha \) for every \( x \in X, a \in G \). If \( f \in L(G), x \in X, \phi \) and \( \psi \in H \), we have: \( (A_j T_x \phi, \psi) = (A_f \phi, \psi) = \int f(a) (U_\alpha T_x \phi, \psi) da = \int f(a) U_\alpha T_x \phi, \psi) da = \int (T_{f(a)} U_\alpha \phi, \psi) da \). Hence \( A_f \phi \) is given by the Pettis integral \( \int T_{f(a)} U_\alpha \phi da \) because \( H \) is its own conjugate space.

Suppose now that \( g \) is an arbitrary function in \( B(G, X) \). Since the finite linear combinations of functions of the type \( f x \) with \( f \in L(G), x \in X \) are dense in \( B(G, X) \), we have a sequence \( g_n \to g \) in \( B(G, X) \) -norm such that \( (A_{g_n} \phi, \psi) = \int (T_{g_n(a)} U_\alpha \phi, \psi) da \) and \( (A_{g_n} \phi, \psi) \to (A_g \phi, \psi) \) as \( n \to \infty \). We show that \( (A_g \phi, \psi) = \int (T_{g(a)} U_\alpha \phi, \psi) da \) and this will establish the first part of the theorem. We have:
We turn to the converse of the first part of the theorem. For any \( f \in B(G, X) \) there is a sequence \( \{g_n\} \) of simple functions converging to \( f \) strongly, almost everywhere. Each function \((Tg_n(a)U\alpha\psi, \phi)\), for any \( \phi, \psi \in H \), is measurable in \( a \) because \( \{U\alpha\} \) is weakly continuous. Since \((Tg_n(a)U\alpha\psi) \rightarrow (Tf(a)U\alpha\psi)\) a.e. as \( n \to \infty \), it follows that \((Tf(a)U\alpha\psi)\) is measurable in \( a \) being the pointwise limit of measurable functions.

Now, since \((Tf(a)U\alpha\psi)\) is measurable, we may speak of the integral \( \int (Tf(a)U\alpha\psi) \, da \). This may be viewed as a bounded bilinear functional in \( H \) for each \( f \in B \), for \( \| \int (Tf(a)U\alpha\psi) \, da \| \leq K \| f \|_B \cdot \| \phi \| \cdot \| \psi \| \). Hence there exists a unique operator \( A_f \) in \( H \), for each \( f \in B \), such that \((Af\phi, \psi) = \int (Tf(a)U\alpha\psi) \, da \) and \( \| A_f \| \leq K \| f \|_B \). We show \( \{A_f\} \) is a representation of \( B(G, X) \). It is clear that \( A_{f+g} = A_f + A_g \) and \( A_{\lambda f} = \lambda A_f \). To show \( A_{f g} = A_f A_g \), we first observe that \((Tf(a)U\alpha\phi, \psi) = \int (Tf(b)T\alpha(b)U\alpha\psi) \, db \) for any \( f \in B \) and \( \phi, \psi \in H \). This latter equation is true for simple functions and the general result follows from the usual approximation argument. Therefore, for any \( f, g \in B \) and \( \phi, \psi \in H \), \( (Tf(a)Tg(a-b)\phi, \psi) = \int (Tf(b)Tg(a-b)\phi, \psi) \, db \) a.e. in \( a \). Now:

\[
(A_{f g} \phi, \psi) = \int (Tf(a)U\alpha\phi, \psi) \, da = \int \int (Tf(b)Tg(a-b)U\alpha\phi, \psi) \, db \, da
\]

\[
= \int \int (Tf(b)Tg(a-b)U\alpha\phi, \psi) \, db \, da
\]

\[
= \int \int (Tf(b)Tg(a)U\alpha\phi, \psi) \, db \, da
\]

\[
= \int \int (UaTg(a)\phi, (UbTf(b))*\psi) \, db \, da
\]

\[
= \int (UbTf(b)A\alpha\phi, \psi) \, db = (A_f A_g \phi, \psi).
\]

Since \( \phi, \psi \) are arbitrary, it follows that \( A_{f g} = A_f A_g \). In several steps,
in the above computation, we use the assumption that the \( T_x \) commute with the \( U_a \) and, in one step, we employed the Fubini theorem.

The last part of the theorem remains to be established. Suppose \( \{ T_x \} \) is a \(*\)-representation of \( X \) in \( H \). Then \( A \bar{\phi} = A^*_\phi \) as we show:

\[
(A \bar{\phi}, \psi) = \int (T_{f(a)} U_a \phi, \psi) \, da = \int (T_{f(-a)}^* U_a \phi, \psi) \, da
\]

\[
= \int (T_{f(-a)}^* U_a \phi, \psi) \, da = \int (T_{f(a)}^* U_a \phi, \psi) \, da
\]

\[
= \int ((U_a T_f(a))^* \phi, \psi) \, da = \left[ \int (U_a T_f(a) \psi, \phi) \, da \right]^*
\]

\[
= (A \psi, \phi)^* = (A^*_\phi, \psi).
\]

Hence \( A \bar{\phi} = A^*_\phi \) and this completes the proof.

Let \( \mathfrak{M}(X) \) denote the space of maximal ideals in \( X \) in the usual weak topology and let \( \hat{G} \) denote the dual (character) group of \( G \) in the Pontrjagin topology. Suppose \( \phi_M \) is the canonical homomorphism of \( X \) onto the complex numbers associated with \( M \in \mathfrak{M}(X) \) and define the "Fourier transform" of \( f \in B \) as \( \hat{f}(M, \delta) = \int \phi_M f(a)(a, \delta) \, da \). The transform \( \hat{f} \) is a function of both \( M \) and \( \delta \). It has been shown [2i] that each continuous multiplicative linear functional in \( B(G, X) \) is of the "Fourier transform" type and, further, \( \mathfrak{M}(B) \) (the space of regular maximal ideals in \( B(G, X) \)) is homeomorphic with the topological product of \( \mathfrak{M}(X) \) and \( \hat{G} \). We use these results from [2i] in proving

**Theorem 2 (Generalized Stone theorem).** Suppose \( X \) is a self-adjoint algebra and \( \{ T_x \}, \{ U_a \}, G \) and \( H \) have the meanings of Theorem 1. Let \( \{ T_x \} \) be a \(*\)-representation commuting with \( \{ U_a \} \). Then there exists a projection-valued measure \( P(M, \delta) \) on \( \mathfrak{M}(X) \times \hat{G} \) (see above) such that

\[
(T_x U_a \phi, \psi) = \int_{\mathfrak{M}(X) \times \hat{G}} \phi_M(x)(a, \delta) d(P(M, \delta) \phi, \psi)
\]

for all \( a \in G, x \in X, \phi \) and \( \psi \in H \).

**Proof.** Since \( \{ T_x \} \) is a \(*\)-representation of \( X \), \( \{ T_x \} \) is a bounded representation. By Theorem 1, \( A \phi = \int T_{f(a)} U_a \phi \, da \) defines a \(*\)-representation of \( B(G, X) \) in \( H \). It is known, [7, Theorem 57], that if \( \{ S_y \} \) is a \(*\)-representation of a self-adjoint algebra \( Y \) by operators in a Hilbert space \( H \), then there exists a projection-valued measure \( P(M) \) defined on the space \( \mathfrak{M}(Y) \) of regular maximal ideals in \( Y \).
such that \((S_P \phi, \psi) = \int_{\mathfrak{M}(X)} \phi_M(y) d(P(M) \phi, \psi)\) for \(\phi, \psi \in H\). Taking \(Y = B(G, X)\) and \(\{S_P\}\) to be \(\{A_f\}\), we have a projection-valued measure \(P(M, \delta)\) defined over \(\mathfrak{M}(X) \times \hat{G}\) such that

\[
(A_f \phi, \psi) = \int (T_f U_{\delta} \phi, \psi) d\delta = \int -\int \bar{f}(M, \delta) d(P(M, \delta) \phi, \psi).
\]

(\(\int -\) means integrate over \(\mathfrak{M}(X) \times \hat{G}\).)

Suppose \(g \in L(G), x \in X\). Then

\[
(A_{g\delta} \phi, \psi) = \int g(a) (T_x U_{\delta} \phi, \psi) d\delta
= \int -\left( \int g(a) \phi_M(x)(a, \delta) d\delta \right) d(P(M, \delta) \phi, \psi)
= \int g(a) \left( \int -\phi_M(x)(a, \delta) d(P(M, \delta) \phi, \psi) \right) d\delta.
\]

Since \(g\) is arbitrary in \(L(G)\) it follows that

\[
(T_x U_{\delta} \phi, \psi) = \int -\phi_M(x)(a, \delta) d(P(M, \delta) \phi, \psi).
\]

This ends the proof.

We conclude this section with two observations. First, if \(G\) reduces to the one-point group \(\{0\}\) and \(X\) is arbitrary, then \(\{U_a\}\) reduces to the single identity operator \(I\) in \(H\) and the above result becomes \((T_x \phi, \psi) = \int_{\mathfrak{M}(X)} \phi_M(x) d(P(M) \phi, \psi)\) which is precisely Theorem 57 in [7]. Second, if \(G\) is arbitrary while \(X\) reduces to the algebra of complex numbers and \(T_x = x I\), then

\[
(x U_a \phi, \psi) = x(U_a \phi, \psi) = \int_\hat{G} x(a, \delta) d(P(\delta) \phi, \psi)
= x \int_\hat{G} (a, \delta) d(P(\delta) \phi, \psi)
\]

so that

\[
(U_a \phi, \psi) = \int_\hat{G} (a, \delta) d(P(\delta) \phi, \psi).
\]

Our theorem thus specializes into Stone's theorem.

3. Kernels and hulls. In this section we shall be concerned with proving a few results on kernels and hulls in \(B(G, X)\). We will apply
these to the (classical) group algebra \( L(G_1 \times G_2) \) over the topological product of two locally compact abelian groups. We do not assume that \( X \) possesses an identity except in Theorem 4.

We recall that if \( X \) is a \( B \)-algebra and \( A \) is a closed ideal in \( X \), then the hull of \( A \), \( h(A) \), is the set of all regular maximal ideals containing \( A \). Let \( k(h(A)) \) be the kernel of \( h(A) \), i.e., the intersection of all ideals in \( h(A) \). An ideal \( A \) is said to be "the kernel of its hull" if \( A = k(h(A)) \).

If \( A \) is an ideal in \( X \) then \( I(A) \) shall denote

\[
\{ f \in B(G, X) \mid f(a) \in A \text{ a.e. over } G \}.
\]

If \( A \) is a proper closed ideal in \( X \), then \( I(A) \) is a proper closed ideal in \( B(G, X) \) as is easy to show.

As indicated in §2, each \((M, \delta) \in \mathcal{M}(X) \times \hat{G}\) gives rise to a multiplicative linear functional in \( B \). We shall denote the regular maximal ideal in \( B \) which is the kernel of this functional by \( I(M, \delta) \).

**Theorem 3.** (i) If \( A \) is a proper closed ideal in \( X \), then \( h(I(A)) = \{ I(M, \delta) \mid M \in h(A), \delta \in \hat{G} \} \).

(ii) \( k(hI(A)) = I(A) \) if and only if \( k(h(A)) = A \).

**Proof.** (i) Suppose \( f \in I(A) \) and \( M \in h(A) \). Then for any \( \delta \in \hat{G} \) we have \( \int \phi_M f(a)(a, \delta) da = 0 \) so that \( I(M, \delta) \in h(I(A)) \).

Assume that \( I(N, \eta) \supset I(A) \) for some \( (N, \eta) \in \mathcal{M}(X) \times \hat{G} \). We wish to show that \( N \in h(A) \). Let \( f = gx \) with \( g \) chosen in \( L(G) \) such that \( g(b) = \int g(a)(a, b) da \neq 0 \) and with \( x \in A - \{0\} \). Clearly \( gx \in I(A) \). Now, \( \int \phi_N (gx)(a)(a, b) da = \int \phi_N (x) g(a)(a, b) da = g(\delta) \phi_N (x) = 0 \). This means \( \phi_N (x) = 0 \) for each \( x \in A \). Hence \( N \supset A \) and \( N \in h(A) \).

(ii) Suppose \( A = k(h(A)) \). If \( f \in k(hI(A)) \), then \( \int (M, \delta) = 0 \) for all \( M \in h(A) \) and all \( \delta \in \hat{G} \). Since \( L(G) \) is semi-simple, \( \phi_M f = 0 \) a.e. over \( G \) for all \( M \in h(A) \). This means \( f \in k(h(A)) \) a.e. over \( G \), as we proceed to show. If \( \{ j_w \} \) is an approximate identity in \( L(G) \) consisting of bounded functions, then \( j_w f \) is continuous from \( G \) to \( X \) since \( f \) is continuous in \( B \)-norm. Hence \( j_w f \in k(h(A)) \) everywhere over \( G \). Let \( \{ j_{w_n} \} \) be a sequence chosen from \( \{ j_w \} \) such that \( j_{w_n} f \to f \) in \( B \)-norm. There exists a subsequence \( \{ j_{w_{n_k}} \} \) in \( \{ j_{w_n} \} \) such that \( j_{w_{n_k}} f \to f \) pointwise a.e. in \( X \)-norm. Hence \( f \in k(h(A)) \) a.e. over \( G \).

Suppose, to prove the converse, that \( k(h(A)) \neq A \). There exists an \( x \in k(h(A)) \) such that \( x \notin A \). The function \( gx \), with \( g \neq 0 \) in \( L(G) \), is in \( k(hI(A)) \) but not in \( I(A) \).

**Corollary.** (1) The only regular maximal ideals containing \( I(M) \) with \( M \in \mathcal{M}(X) \) are of the form \( I(M, \delta) \) with \( \delta \in \hat{G} \). Also \( I(M) \) is the kernel of its hull.
(2) If \( A \) is a closed ideal in \( X \) not contained in any regular maximal ideal, then \( I(A) \) is not contained in any regular maximal ideal in \( B(G, X) \).

(3) There exists an ideal in \( B(G, X) \) which is not the kernel of its hull if this is true of the algebra \( X \).

(4) If \( M, N \in \mathfrak{M}(X) \) with \( M \not= N \) and if \( \delta \in \hat{G} \), then \( I(M, \delta) + I(N) = B(G, X) \).

**Proof.** (1) follows from Theorem 3 (i), (ii). (2) and (3) follow from Theorem 3, parts (i) and (ii), respectively. To prove (4), we observe that \( J = I(M, \delta) + I(N) \) is an ideal containing \( I(M, \delta) \) and \( I(N) \). This means \( J = I(M, \delta) \) or \( J = B(G, X) \). If \( J = I(M, \delta) \), then \( I(M, \delta) \subseteq I(N) \) which is impossible by (1) of the corollary. Hence \( J = B(G, X) \).

We apply the above results to the algebra \( L(G_1 \times G_2) \) where \( G_1 = \{a\} \), \( G_2 = \{b\} \) are two locally compact abelian groups. If \( f \in L(G_1 \times G_2) \) and if \( a \in G_1 \), then the function \( f_a \) defined over \( G_2 \) by \( f_a(b) = f(a, b) \) is designated as the section of \( f \) determined by \( a \). Since \( f \in L(G_1 \times G_2) \), every section of \( f \) is measurable over \( G_2 \) and almost every section \( f_a \) is in \( L(G_2) \) [1, p. 148]. The map \( \pi: L(G_1 \times G_2) \to B(G_1, L(G_2)) \) defined as \( \pi f(a) = f_a(\cdot) \) can be easily shown to be an isometric isomorphism. By means of the map \( \pi \), \( L(G_1 \times G_2) \) may be viewed as an algebra \( B(G, X) \) with \( G = G_1 \) and \( X = L(G_2) \). Hence we may interpret our preceding results in the language of \( L(G_1 \times G_2) \).

The ideals \( I(M) \) then become ideals \( I(b) \) consisting of those functions \( f \in L(G_1 \times G_2) \) such that \( \int G_2 f(a, b)(b, \delta)b\delta = 0 \) a.e. in \( a \) for a fixed \( b \in \hat{G}_2 \). The \( I(M, \delta) \) become \( I(b, \delta) \). These consist of those functions \( f \in L(G_1 \times G_2) \) such that \( \int \int G_1 G_2 f(a, b)(a, \delta)(b, \delta)b\delta a\delta = 0 \).

Corollary 1 to Theorem 3 states that the ideals \( I(b) \) in \( L(G_1 \times G_2) \) are the kernels of their hulls.

If \( G_2 \) is taken to be the additive group of Euclidean \( n \)-space \( E_n \) \((n \geq 3)\), then L. Schwartz has shown [8] that there are closed ideals in \( L(E_n) \) which are smaller than the kernels of their hulls. Hence Theorem 3, Corollary 3 says that the group algebras \( L(G \times E_n) \) have closed ideals smaller than the kernels of their hulls for any abelian group \( G \) when \( n \geq 3 \).

As a further application we see that \( I(b_1, \delta) + I(b_2) = L(G_1 \times G_2) \) for any \( \delta \in \hat{G}_1 \) and \( b_1, b_2 \in \hat{G}_2 \) with \( b_1 \neq b_2 \), by (4) of the Corollary.

We shall return to \( L(G_1 \times G_2) \) after some more results about \( B(G, X) \) are established. First we observe that the Segal-Kaplansky-
Helson theorem \([3; 6]\) is, in general, no longer true for \(B(G, X)\). This theorem states that a closed ideal in the group algebra \(L(G)\) is the kernel of its hull if the hull has a boundary in \(\hat{G}\) containing no non-null perfect set. As a counter-example for an algebra \(B(G, X)\), let \(G = \{0\}\) be the one-point group; then \(\hat{G} = \{\hat{0}\}\) and \((0, \hat{0}) = 1\). Take \(X\) to be a \(B\)-algebra in which there is a closed primary ideal \(A\), i.e., \(A\) is contained in and smaller than precisely one regular maximal ideal \(M_0\). Then \(h(I(A)) = \{I(M_0, \hat{0})\}\) and \(I(A)\) thus has a one-point hull. Yet \(k(h(A)) \neq I(A)\) by Theorem 3 (ii). We may even take \(X\) in this example to be semi-simple and regular so that these conditions are not sufficient for a general theorem. We can, however, prove:

**Theorem 4.** Let \(X\) have an identity \(e\) and let \(I\) be a closed ideal in \(B(G, X)\) with \(h(I) = \{I(M_i, d_i) \mid i \in \Omega\}\). Suppose the set \(\{d_i \mid i \in \Omega\}\) has a boundary in \(\hat{G}\) which contains no non-null perfect set and suppose \(I \supset I(M_0)\) for some \(M_0 \in \mathfrak{M}(X)\). Then \(k(h(I)) = 1\).

**Proof.** Since \(I \supset I(M_0)\) for some \(M_0 \in \mathfrak{M}(X)\), we have \(M_i = M_0\) for all \(i \in \Omega\) by Theorem 3, Corollary (1). Therefore \(h(I) = \{I(M_0, d_i) \mid i \in \Omega\}\). Define \(K = \{f \in B(G, X) \mid f \in I \cap L_e\}\). \(K\) is seen to be a closed ideal in \(L_e\) and it may be shown that \(h(K)\) (the hull of \(K\) in the \(B\)-algebra \(L_e\)) is \(\{d_i \mid i \in \Omega\}\). By hypothesis, the boundary of \(h(K)\) contains no non-null perfect set and this means \(K = k(h(K))\) by the Segal-Kaplansky-Helson theorem. This, in turn, implies \(I = k(h(I))\).

For, let \(f \in k(h(I))\). Then \(f(M_0, d_i) = 0\) for all \(d_i \in h(K)\) and so \((\phi_{M_0})e \in K\). Since \(f - (\phi_{M_0})e \in I(M_0)\) and \(I \supset I(M_0)\), we have \(f = (\phi_{M_0})e + [f - (\phi_{M_0})e]e \in I\).

The next theorem will pertain to compact groups and will require the following lemma.

**Lemma.** Suppose \(\phi_{Mf}\) is constant a.e. for each \(M \in \mathfrak{M}(X)\) where \(f \in B(G, X)\) with \(G\) compact and \(X\) semi-simple. Then \(f\) is constant a.e.

**Proof.** Since \(\phi_{Mf}\) is constant a.e. and since \(f(a, d)da = 0\) if \(d \neq \hat{0}\), we have \(f(M, d) = 0\) if \(d \neq \hat{0}\) for all \(M \in \mathfrak{M}(X)\). If the Haar measure of \(G\) is normalized to 1, then \(f(M, d) - f(M, \hat{0}) \cdot f(a, d)da = 0\) for all \((M, d)\) because \(f(a, \hat{0})da = 1\). This means \(f - f(a)da\) has a Fourier transform vanishing everywhere over \(\mathfrak{M}(X) \times \hat{G}\). Thus \(f = \mathfrak{F}f(a)da\) a.e. over \(G\) because \(B(G, X)\) is semi-simple if \(X\) is semi-simple \([2i]\).

**Theorem 5.** Suppose \(G\) is a compact group (not the one-point group). Then \(X\) is isometrically and isomorphically contained in \(B(G, X)\) and \(X\) is a closed ideal in \(B(G, X)\). Further, \(X\) is the kernel of its hull if and only if \(X\) is semi-simple as a \(B\)-algebra.
Proof. We may consider the $B$-algebra $X$ as contained as a subset of $B(G, X)$ by identifying each $x \in X$ with the constant mapping $G \to x$. The elements of $X$ can thus be viewed as constants in $B(G, X)$. When the Haar measure $m$ of $G$ is normalized so that $m(G) = 1$, then $X$ is isometrically and isomorphically contained in $B(G, X)$ as is readily seen. If $g \in B$, $x \in X$, then $g \ast x = g(0)x$ so that $X$ is a closed ideal in $B$.

Now $I(M, \hat{0}) \supset X$ for all $(M, \hat{0})$ with $\hat{0} \neq \hat{0}$. We will prove that $I = \cap I(M, \hat{0})$, where the intersection is taken over $\pi(X) \times (\hat{G} - \{\hat{0}\})$, is identical with $X$. If $f \in I$, then $f(M, \hat{0}) = 0$ for all $\hat{0} \neq \hat{0}$ so that $\phi_Mf$ is a constant a.e. for all $M \in \pi(X)$. Since $X$ is semi-simple, $f$ is constant a.e. by the preceding lemma. $X$ is therefore the kernel of its hull.

Suppose $X$ is not semi-simple. If $f$ is not a constant function in $L(G)$ (such a function exists if $G$ is not the one-point group) and if $x(\neq 0)$ is in all the regular maximal ideals of $X$, then $fx \in I$ (see above), but $fx \notin X$.

We conclude with more applications to $L(G \times G_2)$. If $G_2$ is taken to be a discrete group, then $L(G_2)$ has an identity and Theorem 4 is applicable. We construct an ideal $I$ in $L(G_1 \times G_2)$ which meets the requirements of this theorem. Suppose $\{\hat{a}_i\}$ is a set in $\hat{G}_1$ with boundary containing no non-null perfect set. For example, $\{\hat{a}_i\}$ may be taken finite. Let $f$ be a function in $L(G_1)$ whose Fourier transform vanishes precisely at the set $\{\hat{a}_i\}$. Suppose $b_1, b_2 \in G_2$ with $b_1 \neq b_2$. Choose $g \in L(G_2)$ such that $\hat{g}(\hat{b}_1) \neq 0$, $\hat{g}(\hat{b}_2) = 0$. Let $I$ be the closed ideal generated by $I(\hat{b}_1)$ and the function $f(\cdot)g(\cdot) \in L(G_1 \times G_2)$, i.e., $I$ consists of all limits of sequences of functions of the type $f(\cdot)g(\cdot) * r(\cdot, \cdot) + s(\cdot, \cdot)$ with $r \in L(G_1 \times G_2)$ and $s \in I(\hat{b}_1)$. The hull of $I$ is $\{I(\hat{b}_1, \hat{a}_i)\}$ as is easy to show. $I$ is therefore the kernel of its hull by Theorem 4.

Now consider $L(G_1 \times G_2)$ with $G_1$ a compact group and $G_2$ arbitrary. Since $L(G_2)$ is semi-simple, Theorem 5 says that the functions $f \in L(G_1 \times G_2)$, which are independent of $a \in G_1$, form a closed ideal which is the kernel of its hull.

Bibliography


Let $L$ be a Lie algebra over a ground field of characteristic 0 and let $D(L)$ and $I(L)$ denote the Lie algebra of derivations and inner derivations of $L$ respectively.

The following theorem is proved, although not explicitly stated, in an earlier note.\textsuperscript{1}

**Theorem 1.** Let $L = S + R$ (Levi decomposition) and let $\mathfrak{H}(S) = \{D \mid D \subseteq D(L), D(S) = (0)\}$. Further let $\rho$ denote the restriction homomorphism of $D(L)$ into $D(R)$. Then $D(L)$ splits over $I(L)$ if and only if $\rho(\mathfrak{H}(S))$ splits over $\rho(\mathfrak{H}(S)) \cap I(R)$.

If $S$ is a semi-simple Lie algebra and $M$ is any $S$-module then $M$ is the direct sum of $M^S$ and $S \cdot M$ where $M^S$ is the trivial submodule of $M$.

**Theorem 2.** Let $L = S + R$ (Levi decomposition). If $R^S \subseteq Z(R)$ (the center of $R$) then $D(L)$ splits over $I(L)$\textsuperscript{2}.

**Proof.** Let $u \in R$ and suppose that the derivation of $R$ that is effected by $u$ is the restriction to $R$ of a derivation of $L$ that annihilates $S$. Then $[S, u] \subseteq Z(R)$. As an $S$-module, $R$ is the direct sum of $Z(R)$ and a complementary submodule $P$. The component of $u$ in $P$ is annihilated by $S$ so that $u \in R^S + Z(R) \subseteq Z(R)$. This means

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\textsuperscript{1} Details will be found in G. F. Leger, A note on the derivations of Lie algebras, Proc. Amer. Math. Soc. vol. 4 (1953). We refer to this note as I.

\textsuperscript{2} The author is grateful to the referee for putting Theorem 2 in this form.