A MAXIMUM MODULUS PROPERTY OF MAXIMAL SUBALGEBRAS

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In a recent paper [6] Wermer considered the algebra C of all continuous complex valued functions on γ, a simple closed analytic curve bounding a region Γ, with $\Gamma\cup\gamma$ compact, on a Riemann surface F. He considered the subalgebra A of all functions in C which could be extended into Γ to be analytic on Γ and continuous on $\Gamma\cup\gamma$. Wermer showed that A was a maximal closed subalgebra of C which separated the points of γ, and that the space of maximal ideals of A was homeomorphic to $\Gamma\cup\gamma$.

In [2] Civin and Yood considered a class of subalgebras of complex commutative regular Banach algebras which become maximal closed subalgebras in the event the original algebra was the collection of continuous functions on a compact Hausdorff space. The object of this note is to demonstrate that such subalgebras possess a maximum modulus property possessed by A. To state the result obtained we recall certain definitions. The terms not herein defined may be found in [5].

Let B be a complex commutative regular Banach algebra with identity e and space of maximal ideals $\mathfrak{M}(B)$. Let $\pi: x\rightarrow x(M)$ be the Gelfand representation of B as a subalgebra of $C(\mathfrak{M}(B))$, the continuous function on $\mathfrak{M}(B)$. We also denote πx by $\hat{x}$ and πQ by $\hat{Q}$ for any subset Q of B. A subalgebra N of B is called *determining* [2] if πN is dense in πB, otherwise N is called *nondetermining*. A subalgebra of B is called a *maximal nondetermining* subalgebra if every larger subalgebra of B is deterning. A subset S of B is called a *separating family* on $\mathfrak{M}(B)$ if for each M_1, M_2 in $\mathfrak{M}(B)$, $M_1\neq M_2$, there exists an $x\in S$ such that $x(M_1)\neq x(M_2)$. If P is an algebra of continuous complex valued functions vanishing at infinity on the locally compact space X, the smallest closed set (if it exists) on which each $|f|$ with $f\in P$ assumes its maximum is called the *Šilov boundary of X with respect to P.*

THEOREM 1. *Let B be a complex commutative regular Banach algebra with identity e, and let N be a maximal nondetermining subalgebra of*

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B which is not a maximal ideal. If N is a separating family on $\mathfrak{M}(B)$, then $\mathfrak{M}(B)$ may be topologically embedded ifi $\mathfrak{M}(N)$ and as so embedded $\mathfrak{M}(B)$ is the Šilov boundary of $\mathfrak{M}(N)$ with respect to N.

While the present note was in the process of publication, two proofs of Theorem 1 appeared for the special case when $B=C(X)$ for a compact Hausdorff space X, one by H. S. Bear [1] and the other by K. Hoffman and I. M. Singer [**4**].

Before proceeding to the proof of the theorem, we require one lemma, which was noted by Helson and Quigley [3] for the case $B=C(X)$.

LEMMA 2. *Let N be a maximal nondetermining subalgebra of the complex commutative regular Banach algebra B, and let e be the identity of B. Then either $e\in N$ or N is a maximal ideal of B.*

Suppose $e\notin N$. Let $D=\{a+\lambda e: a\in N$ and λ complex$\}$. As e is the unit for B, D is a subalgebra of B which properly contains N, hence $\hat{D}$ is dense in $\hat{B}$. Let $x\in B$ and $a\in N$. There exists $a_n\in N$ and λ_n complex, $n=1, 2, \cdots$, such that $\pi(a_n+\lambda_n e)\rightarrow\pi x$ as in $n\rightarrow\infty$. Therefore $(a_n+\lambda_n e)a\in N$ and $\pi\{(a_n+\lambda_n e)a\}\rightarrow\pi(xa)$. By Lemma 1 of [**2**], $\hat{N}$ is closed in $\hat{B}$, so $\pi(xa)\in\hat{N}$. There thus exists $u\in N$ such that $xa-u$ is in the radical of B. As noted in [**2**], N contains the radical of B. Thus $xa\in N$ and N is an ideal of B. That N is a maximal ideal is an immediate consequence of N being maximal nondetermining.

We return to the proof of Theorem 1. Each nonzero multiplicative linear functional on B is automatically one on N, and distinct multiplicative linear functionals on B have distinct restrictions to N since N is a separating family on $\mathfrak{M}(B)$. There is thus a one-to-one correspondence between $\mathfrak{M}(B)$ and a subset of $\mathfrak{M}(N)$. The mapping is clearly continuous from $\mathfrak{M}(B)$ to $\mathfrak{M}(N)$ in the Gelfand topologies. As $\mathfrak{M}(B)$ is a compact Hausdorff space, the mapping is a homeomorphism. We henceforth suppose $\mathfrak{M}(B)$ is a subset of $\mathfrak{M}(N)$.

Since N is a subalgebra of B, $\lim \|a^n\|^{1/n}$ is independent of whether the N or B norm is used. Thus $\sup |a(M)|$ is the same whether calculated over $\mathfrak{M}(B)$ or $\mathfrak{M}(N)$. To see that $\mathfrak{M}(B)$ is the Šilov boundary of $\mathfrak{M}(N)$ with respect to N, it is sufficient to see that there is no proper closed subset of $\mathfrak{M}(B)$ on which each $|a(M)|$, $a\in N$, attains its maximum. Suppose otherwise and let $\mathfrak{K}$ be a proper closed subset of $\mathfrak{M}(B)$ of the required type.

Let $M_0\in\mathfrak{M}(B)$, $M_0\notin\mathfrak{K}$. If $\mathfrak{L}$ is any closed set in $\mathfrak{M}(B)$ such that $\mathfrak{L}\supset\mathfrak{K}$ and $M_0\notin\mathfrak{L}$, let $\mathfrak{B}$ be an open set in $\mathfrak{M}(B)$ with $M_0\in\mathfrak{B}$ and $\overline{\mathfrak{B}}\cap\mathfrak{L}=\overline{\mathfrak{B}}$, the closure being in $\mathfrak{M}(B)$. Let $W=W(\mathfrak{L})$ be the kernel of $\mathfrak{L}$, i.e. $W=\bigcap M$, $M\in\mathfrak{L}$. Let R be the radical of B. Since B is a regular

Banach algebra, W contains elements not in R. Consider the algebra $S=N+W$. The elements of S are of the form $a+u$, $a\in N$, $u\in W$, since W is an ideal of B. For $u\in W$, $u\notin R$, the maximum modulus of $u(M)$ is not attained on $\mathfrak{K}$, so $u\notin N$, and thus S contains N properly. As N was maximal nondetermining, $\hat{S}$ is dense in $\hat{B}$.

Let $b\in B$. There exists $a_n\in N$, $u_n\in W$, $n=1, 2, \cdots$, so that if $r_n=a_n+u_n$, then $\hat{r}_n\to\hat{b}$. For $M\in\mathfrak{L}$, $|a_n(M)-a_m(M)|=|r_n(M)-r_m(M)|$. Thus

$$\sup_{M\in\mathfrak{L}} |a_n(M) - a_m(M)| \leqq \sup_{M\in\mathfrak{M}(B)} |r_n(M) - r_m(M)|.$$

By the maximum modulus property of N relative to $\mathfrak{K}\subset\mathfrak{L}$,

$$\sup_{M\in\mathfrak{M}(B)} |a_n(M) - a_m(M)| \leqq \sup_{M\in\mathfrak{M}(B)} |r_n(M) - r_m(M)|.$$

Since $\hat{N}$ is closed [2], there exists $a_0\in N$ such that $\hat{a}_n\to\hat{a}_0$. There is then an element $w_0\in W$ such that $\hat{u}_n\to\hat{w}_0$. If $b_0=a_0+w_0$, $\hat{r}_n\to\hat{b}$ and $\hat{r}_n\to b_0$, and consequently $\hat{b}-\hat{b}_0=0$ and $b-b_0\in R$. As noted in [2], $R\subset N$, so $b-b_0\in N$. Since b was arbitrary, $B=N+W=N+W(\mathfrak{L})$.

We next show the complement of $\mathfrak{K}$ in $\mathfrak{M}(B)$ consists of a single point. Suppose otherwise. Let $M_i\in\mathfrak{M}(B)$, $M_i\notin\mathfrak{K}$, $i=1, 2$, and $M_1\neq M_2$. Let $\mathfrak{L}$ be a closed set in $\mathfrak{M}(B)$, such that $\{M_1\}\cup\mathfrak{K}\subset\mathfrak{L}$ and $M_2\notin\mathfrak{L}$. Since B is a regular Banach algebra, there is an element $b\in B$, such that $b(M)=0$, $M\in\mathfrak{K}$, and $b(M_1)=1$. We may express b as $b=a+u$, $a\in N$, $u\in W(\mathfrak{L})$. For $M\in\mathfrak{K}$, $0=b(M)=a(M)+u(M)$. Since $u(M)=0$ for $M\in\mathfrak{L}$, $a(M)=0$ for $M\in\mathfrak{K}$. However, $1=b(M_1)=a(M_1)+u(M_1)=a(M_1)$ since $M_1\in\mathfrak{L}$. This contradicts the supposition that for $a\in N$,

$$\sup_{M\in\mathfrak{K}} |a(M)| = \sup_{M\in\mathfrak{M}(B)} |a(M)|.$$

Thus $\mathfrak{M}(B)=\mathfrak{K}\cup\{M_0\}$, and since $\mathfrak{K}$ was closed in $\mathfrak{M}(B)$, M_0 is an isolated point of $\mathfrak{M}(B)$.

Let $W=W(\mathfrak{K})=\{f\in B\,|\,\hat{f}(\mathfrak{K})\equiv 0\}$. Consider any element $b+W$ of B/W. Since $b=a+u$, with $a\in N$, $u\in W$, there is an element a of N in the coset $b+W$. Now $R\subset W$, so all elements of the coset $a+R$ of N/R are in the coset $b+W$. Moreover if $a_i\in b+W$, and $a_i\in N$, $i=1, 2$, then $a_1-a_2\in W$ so by the maximum modulus property that $\mathfrak{K}$ is alleged to have $\hat{a}_1-\hat{a}_2=0$ and $a_1-a_2\in R$. There is thus a one-to-one correspondence between the cosets $b+W$ and $a+R$. The correspondence gives an isomorphism of B/W and N/R.

Let $N_1=\{a\in N: a(M_0)=0\}$. Then N_1 is a maximal ideal of N which contains R and thus N_1/R is a maximal ideal of N/R. The isomorphism obtained above implies the existence of a maximal ideal M_1 in B, $M_1 \supset W$ and with M_1/W isomorphic to N_1/R. Since $M_1 \supset W$, $M_1 \neq M_0$.

Let $a \in N_1$, so $a(M_0)=0$. Then $a(M_1)=0$ because of the inclusion of the coset $a+R$ in the coset $a+W$. Similarly, if $a \in N \cap M_1$, then $a \in M_0$. Lemma 2 implies that for arbitrary $a \in N$, there is a constant λ such that $a-\lambda e \in N_1$. But then $a(M_0)-\lambda = a(M_1)-\lambda$ and N does not separate the points of $\mathfrak{M}(B)$. This contradiction completes the proof of the theorem.

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