A MAXIMUM MODULUS PROPERTY OF MAXIMAL SUBALGEBRAS

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In a recent paper [6] Wermer considered the algebra $C$ of all continuous complex valued functions on $\gamma$, a simple closed analytic curve bounding a region $\Gamma$, with $\Gamma \cup \gamma$ compact, on a Riemann surface $F$. He considered the subalgebra $A$ of all functions in $C$ which could be extended into $\Gamma$ to be analytic on $\Gamma$ and continuous on $\Gamma \cup \gamma$. Wermer showed that $A$ was a maximal closed subalgebra of $C$ which separated the points of $\gamma$, and that the space of maximal ideals of $A$ was homeomorphic to $\Gamma \cup \gamma$.

In [2] Civin and Yood considered a class of subalgebras of complex commutative regular Banach algebras which become maximal closed subalgebras in the event the original algebra was the collection of continuous functions on a compact Hausdorff space. The object of this note is to demonstrate that such subalgebras possess a maximum modulus property possessed by $A$. To state the result obtained we recall certain definitions. The terms not herein defined may be found in [5].

Let $B$ be a complex commutative regular Banach algebra with identity $e$ and space of maximal ideals $\mathcal{M}(B)$. Let $\pi: x \rightarrow x(M)$ be the Gelfand representation of $B$ as a subalgebra of $C(\mathcal{M}(B))$, the continuous function on $\mathcal{M}(B)$. We also denote $\pi x$ by $\hat{x}$ and $\pi Q$ by $\hat{Q}$ for any subset $Q$ of $B$. A subalgebra $N$ of $B$ is called determining [2] if $\pi N$ is dense in $\pi B$, otherwise $N$ is called nondetermining. A subalgebra of $B$ is called a maximal nondetermining subalgebra if every larger subalgebra of $B$ is determining. A subset $S$ of $B$ is called a separating family on $\mathcal{M}(B)$ if for each $M_1, M_2$ in $\mathcal{M}(B)$, $M_1 \neq M_2$, there exists an $x \in S$ such that $x(M_1) \neq x(M_2)$. If $P$ is an algebra of continuous complex valued functions vanishing at infinity on the locally compact space $X$, the smallest closed set (if it exists) on which each $|f|$ with $f \in P$ assumes its maximum is called the Silov boundary of $X$ with respect to $P$.

Theorem 1. Let $B$ be a complex commutative regular Banach algebra with identity $e$, and let $N$ be a maximal nondetermining subalgebra of

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$B$ which is not a maximal ideal. If $N$ is a separating family on $\mathcal{M}(B)$, then $\mathcal{M}(B)$ may be topologically embedded if $\mathcal{M}(N)$ and as so embedded $\mathcal{M}(B)$ is the Silov boundary of $\mathcal{M}(N)$ with respect to $N$.

While the present note was in the process of publication, two proofs of Theorem 1 appeared for the special case when $B = C(X)$ for a compact Hausdorff space $X$, one by H. S. Bear [1] and the other by K. Hoffman and I. M. Singer [4].

Before proceeding to the proof of the theorem, we require one lemma, which was noted by Helson and Quigley [3] for the case $B = C(X)$.

**Lemma 2.** Let $N$ be a maximal nondetermining subalgebra of the complex commutative regular Banach algebra $B$, and let $e$ be the identity of $B$. Then either $e \in N$ or $N$ is a maximal ideal of $B$.

Suppose $e \in N$. Let $D = \{a + \lambda e : a \in N$ and $\lambda$ complex $\}$. As $e$ is the unit for $B$, $D$ is a subalgebra of $B$ which properly contains $N$, hence $\hat{D}$ is dense in $\hat{B}$. Let $x \in B$ and $a \in N$. There exists $a_n \in N$ and $\lambda_n$ complex, $n = 1, 2, \ldots$, such that $\pi(a_n + \lambda_n e) \to \pi x$ as in $n \to \infty$. Therefore $(a_n + \lambda_n e)a \in N$ and $\pi \{(a_n + \lambda_n e)a\} \to \pi(xa)$. By Lemma 1 of [2], $\hat{N}$ is closed in $\hat{B}$, so $\pi(xa) \in \hat{N}$. There thus exists $u \in N$ such that $xa - u$ is in the radical of $B$. As noted in [2], $N$ contains the radical of $B$. Thus $xa \notin N$ and $N$ is an ideal of $B$. That $N$ is a maximal ideal is an immediate consequence of $N$ being maximal nondetermining.

We return to the proof of Theorem 1. Each nonzero multiplicative linear functional on $B$ is automatically one on $N$, and distinct multiplicative linear functionals on $B$ have distinct restrictions to $N$ since $N$ is a separating family on $\mathcal{M}(B)$. There is thus a one-to-one correspondence between $\mathcal{M}(B)$ and a subset of $\mathcal{M}(N)$. The mapping is clearly continuous from $\mathcal{M}(B)$ to $\mathcal{M}(N)$ in the Gelfand topologies. As $\mathcal{M}(B)$ is a compact Hausdorff space, the mapping is a homeomorphism. We henceforth suppose $\mathcal{M}(B)$ is a subset of $\mathcal{M}(N)$.

Since $N$ is a subalgebra of $B$, $\lim ||a^n||^{1/n}$ is independent of whether the $N$ or $B$ norm is used. Thus sup $|a(M)|$ is the same whether calculated over $\mathcal{M}(B)$ or $\mathcal{M}(N)$. To see that $\mathcal{M}(B)$ is the Silov boundary of $\mathcal{M}(N)$ with respect to $N$, it is sufficient to see that there is no proper closed subset of $\mathcal{M}(B)$ on which each $|a(M)|$, $a \in N$, attains its maximum. Suppose otherwise and let $\mathcal{K}$ be a proper closed subset of $\mathcal{M}(B)$ of the required type.

Let $M \in \mathcal{M}(B)$, $M \in \mathcal{K}$. If $\mathcal{L}$ is any closed set in $\mathcal{M}(B)$ such that $\mathcal{L} \supseteq \mathcal{K}$ and $M_0 \in \mathcal{L}$, let $\mathcal{B}$ be an open set in $\mathcal{M}(B)$ with $M_0 \in \mathcal{B}$ and $\overline{\mathcal{B}} \cap \mathcal{L} = \overline{\mathcal{L}}$, the closure being in $\mathcal{M}(B)$. Let $W = W(\mathcal{L})$ be the kernel of $\mathcal{L}$, i.e. $W = \cap M, M \in \mathcal{L}$. Let $R$ be the radical of $B$. Since $B$ is a regular
Banach algebra, $W$ contains elements not in $R$. Consider the algebra $S=N+W$. The elements of $S$ are of the form $a+u$, $a\in N$, $u\in W$, since $W$ is an ideal of $B$. For $u\in W$, $u\in R$, the maximum modulus of $u(M)$ is not attained on $\mathcal{A}$, so $u\notin N$, and thus $S$ contains $N$ properly. As $N$ was maximal nondetermining, $S$ is dense in $B$.

Let $b\in B$. There exists $a_n\in N$, $u_n\in W$, $n=1, 2, \ldots$, so that if $r_n=a_n+u_n$, then $r_n\to b$. For $M\in \mathcal{A}$, $|a_n(M)-a_m(M)|=|r_n(M)-r_m(M)|$. Thus

$$\sup_{M\in \mathcal{A}} |a_n(M)-a_m(M)| \leq \sup_{M\in \mathcal{A}(B)} |r_n(M)-r_m(M)|.$$ 

By the maximum modulus property of $N$ relative to $\mathcal{A}\subset \mathcal{A}$,

$$\sup_{M\in \mathcal{A}(B)} |a_n(M)-a_m(M)| \leq \sup_{M\in \mathcal{A}(B)} |r_n(M)-r_m(M)|.$$ 

Since $N$ is closed [2], there exists $a_0\in N$ such that $a_n\to a_0$. There is then an element $w_0\in W$ such that $a_n\to w_0$. If $b_0=a_0+w_0$, $r_n\to b$ and $r_n\to b_0$, and consequently $b-b_0=0$ and $b-b_0\in R$. As noted in [2], $R\subset N$, so $b-b_0\in N$. Since $b$ was arbitrary, $B=N+W=N+W(\mathcal{A})$.

We next show the complement of $\mathcal{A}$ in $\mathcal{M}(B)$ consists of a single point. Suppose otherwise. Let $M_i\in \mathcal{M}(B)$, $M_i\in \mathcal{A}$, $i=1, 2$, and $M_1\neq M_2$. Let $\mathcal{A}$ be a closed set in $\mathcal{M}(B)$, such that $\{M_1\} \cup \mathcal{A} \subset \mathcal{A}$ and $M_2\in \mathcal{A}$. Since $B$ is a regular Banach algebra, there is an element $b\in B$, such that $b(M)=0$, $M\in \mathcal{A}$, and $b(M_1)=1$. We may express $b$ as $b=a+u$, $a\in N$, $u\in W(\mathcal{A})$. For $M\in \mathcal{A}$, $0=b(M)=a(M)+u(M)$. Since $u(M)=0$ for $M\in \mathcal{A}$, $a(M)=0$ for $M\in \mathcal{A}$. However, $1=b(M_1)=a(M_1)+u(M_1)=a(M_1)$ since $M_1\in \mathcal{A}$. This contradicts the supposition that for $a\in N$,

$$\sup_{M\in \mathcal{A}} |a(M)| = \sup_{M\in \mathcal{A}(B)} |a(M)|.$$ 

Thus $\mathcal{M}(B)=\mathcal{A}\cup \{M_0\}$, and since $\mathcal{A}$ was closed in $\mathcal{M}(B)$, $M_0$ is an isolated point of $\mathcal{M}(B)$.

Let $W=W(\mathcal{A})=\{f\in B| f(\mathcal{A})=0\}$. Consider any element $b+W$ of $B/W$. Since $b=a+u$, with $a\in N$, $u\in W$, there is an element $a$ of $N$ in the coset $b+W$. Now $R\subset W$, so all elements of the coset $a+R$ of $N/R$ are in the coset $b+W$. Moreover if $a_i\in b+W$, and $a_i\in N$, $i=1, 2$, then $a_1-a_2\in W$ so by the maximum modulus property that $\mathcal{A}$ is alleged to have $a_1-a_2=0$ and $a_1-a_2\in R$. There is thus a one-to-one correspondence between the cosets $b+W$ and $a+R$. The corresponding gives an isomorphism of $B/W$ and $N/R$.

Let $N_1=\{a\in N| a(M_0)=0\}$. Then $N_1$ is a maximal ideal of $N$ which contains $R$ and thus $N_1/R$ is a maximal ideal of $N/R$. The iso-
morphism obtained above implies the existence of a maximal ideal $M_i$ in $B$, $M_i \supset W$ and with $M_1/W$ isomorphic to $N_1/R$. Since $M_1 \supset W$, $M_1 \neq M_0$.

Let $a \in N_1$, so $a(M_0) = 0$. Then $a(M_1) = 0$ because of the inclusion of the coset $a + R$ in the coset $a + W$. Similarly, if $a \in N \setminus M_1$, then $a \in M_0$. Lemma 2 implies that for arbitrary $a \in N$, there is a constant $\lambda$ such that $a - \lambda e \in N_1$. But then $a(M_0) - \lambda = a(M_1) - \lambda$ and $N$ does not separate the points of $\mathcal{M}(B)$. This contradiction completes the proof of the theorem.

**Bibliography**


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